The curvilinear motion of a particle involves particle motion along a curved path. The position $P$ of the particle at a given time is defined by the position vector $r$ joining the origin $O$ of the coordinate system with the point $P$.

The velocity of the particle is defined by the relation

$$
\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} \Rightarrow \mathbf{v} = \frac{d\mathbf{r}}{dt}
$$

The velocity vector is tangent to the path of the particle, and has a magnitude (speed) $v$ equal to the time derivative of the length $s$ of the arc described by the particle:

$$
v = \lim_{\Delta t \to 0} \frac{PP'}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \Rightarrow v = \frac{ds}{dt}
$$

Velocity: a vector, speed: a scalar
In general, the acceleration $a$ of the particle is not tangent to the path of the particle. It is defined by the relation $a$ is not second derivative of $s$. 
11.10 DERIVATIVES OF VECTOR FUNCTIONS

\[ P(u) : u \text{ completely defines vector } P. \]

\[ \Delta P = P(u + \Delta u) - P(u) \]

\[ \frac{\Delta P}{\Delta u} = \frac{P(u + \Delta u) - P(u)}{\Delta u} \]

\[ \frac{dP}{du} = \lim_{\Delta u \to 0} \frac{\Delta P}{\Delta u} = \lim_{\Delta u \to 0} \frac{P(u + \Delta u) - P(u)}{\Delta u} \]

As \( \Delta u \) approaches zero, the line of action of \( \Delta P \) becomes tangent to the curve.

The derivative \( \frac{dP}{du} \) of the vector function \( P(u) \) is tangent to the curve described by the tip of \( P(u) \).
Differentiations of sums of vector functions

\[
\frac{d(P + Q)}{du} = \lim_{\Delta u \to 0} \frac{\Delta(P + Q)}{\Delta u} = \lim_{\Delta u \to 0} \left( \frac{\Delta P}{\Delta u} + \frac{\Delta Q}{\Delta u} \right)
\]

\[
\frac{d(P + Q)}{du} = \lim_{\Delta u \to 0} \frac{\Delta P}{\Delta u} + \lim_{\Delta u \to 0} \frac{\Delta Q}{\Delta u}
\]

\[
\frac{d(P + Q)}{du} = \frac{dP}{du} + \frac{dQ}{du}
\]

The product of a scalar function \(f(u)\) and a vector function \(P(u)\) of the same scalar variable \(u\)

\[
\frac{d(fP)}{du} = \lim_{\Delta u \to 0} \frac{(f + \Delta f)(P + \Delta P) - fP}{\Delta u} = \lim_{\Delta u \to 0} \left( \frac{\Delta f}{\Delta u} P + f \frac{\Delta P}{\Delta u} \right)
\]

\[
\frac{d(fP)}{du} = \frac{df}{du} P + f \frac{dP}{du}
\]

\[dU/dt = dU/dx \cdot dx/dt\]
The derivatives of the scalar product and the vector product of two vectors

\[
\frac{d(P \cdot Q)}{du} = \frac{dP}{du} Q + P \frac{dQ}{du} \\
\frac{d(P \times Q)}{du} = \frac{dP}{du} \times Q + P \times \frac{dQ}{du}
\]

The rectangular components of the derivative of a vector function \( \mathbf{P}(u) \)

\[
\mathbf{P} = P_x \mathbf{i} + P_y \mathbf{j} + P_z \mathbf{k} \quad \mathbf{i}, \mathbf{j} \text{ and } \mathbf{k} \text{ have a constant magnitude and fixed directions, derivatives are zero.}
\]

\[
\frac{d\mathbf{P}}{du} = \frac{dP_x}{du} \mathbf{i} + \frac{dP_y}{du} \mathbf{j} + \frac{dP_z}{du} \mathbf{k}
\]

Rate of change of a vector: \( \mathbf{P} \) is a function of time \( t \). \( d\mathbf{P}/dt \): the rate of change of \( \mathbf{P} \) with respect to frame \( Oxyz \).

\[
\frac{d\mathbf{P}}{dt} = \frac{dP_x}{dt} \mathbf{i} + \frac{dP_y}{dt} \mathbf{j} + \frac{dP_z}{dt} \mathbf{k}
\]

\[
\dot{\mathbf{P}} = \dot{P}_x \mathbf{i} + \dot{P}_y \mathbf{j} + \dot{P}_z \mathbf{k}
\]

The rate of change of a vector is the same with respect to a fixed frame and with respect to a frame in translation.
Denoting by $x$, $y$, and $z$ the rectangular coordinates of a particle $P$, the rectangular components of velocity and acceleration of $P$ are equal, respectively, to the first and second derivatives with respect to $t$ of the corresponding coordinates:

The use of rectangular components is particularly effective in the study of the motion of projectiles.
In 2-D, and motion starts from the origin

\[ \nu_x = \dot{x} = (\nu_x)_0, \quad \nu_y = \dot{y} = (\nu_y)_0 - gt, \quad \nu_z = 0 \]

\[ x = (\nu_x)_0 t, \quad y = (\nu_y)_0 t - \frac{1}{2} gt^2, \quad z = 0 \]
We have change or decrease in velocity in the y direction.
We have acceleration (deceleration until the max elevation) and acceleration after max elevation until it hits the ground.

- Motion in horizontal direction is uniform.
- Motion in vertical direction is uniformly accelerated.
- Motion of projectile could be replaced by two independent rectilinear motions.
For two particles \( A \) and \( B \) moving in space, we consider the relative motion of \( B \) with respect to \( A \), or more precisely, with respect to a moving frame attached to \( A \) and in translation with \( A \). Denoting by \( \mathbf{r}_{B/A} \) the relative position vector of \( B \) with respect to \( A \), we have

\[
\frac{d}{dt} \mathbf{r}_{B/A} = \mathbf{v}_{B/A}, \\
\frac{d}{dt} \mathbf{v}_{B/A} = \mathbf{a}_{B/A}.
\]

Denoting by \( \mathbf{v}_{B/A} \) and \( \mathbf{a}_{B/A} \), respectively, the \textit{relative velocity} and the \textit{relative acceleration} of \( B \) with respect to \( A \), we also have

and

\textit{fixed frame of reference.}  \\
\textit{moving frames of reference}
It is sometimes convenient to resolve the velocity and acceleration of a particle $P$ into components other than the rectangular $x$, $y$, and $z$ components. For a particle $P$ moving along a path confined to a plane, we attach to $P$ the unit vectors $\mathbf{e}_t$ tangent to the path and $\mathbf{e}_n$ normal to the path and directed toward the center of curvature of the path.

- $\mathbf{e}_t$ and $\mathbf{e}_t'$ are tangential unit vectors for the particle path at $P$ and $P'$. When drawn with respect to the same origin, $\Delta \mathbf{e}_t = \mathbf{e}_t' - \mathbf{e}_t$ and $\Delta \theta$ is the angle between them.

\[ \Delta e_t = 2 \sin(\Delta \theta/2) \]

\[
\lim_{\Delta \theta \to 0} \frac{\Delta \mathbf{e}_t}{\Delta \theta} = \lim_{\Delta \theta \to 0} \frac{\sin(\Delta \theta/2)}{\Delta \theta/2} \mathbf{e}_n = \mathbf{e}_n
\]

\[ \mathbf{e}_n = \frac{d \mathbf{e}_t}{d \theta} \]
With the velocity vector expressed as the particle acceleration may be written as

\[
\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt} \vec{e}_t + v \frac{d\vec{e}_t}{dt} = \frac{dv}{dt} \vec{e}_t + v \frac{d\vec{e}_t}{d\theta} \frac{d\theta}{ds} \frac{ds}{dt}
\]

After substituting,

\[
\vec{a} = \frac{dv}{dt} \vec{e}_t + \frac{v^2}{\rho} \vec{e}_n \quad a_t = \frac{dv}{dt} \quad a_n = \frac{v^2}{\rho}
\]

Normal acceleration is caused by a physical Connection, e.g. Gravity, cable, friction force if this connection is broken then the normal Acceleration disappears.
The velocity and acceleration are expressed in terms of tangential and normal components. The velocity of the particle is

\[ \mathbf{v} = v \mathbf{e}_t \]

\[ a = \frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}}{dt} \mathbf{e}_t + v \frac{d\mathbf{e}_t}{dt} \]

\[ a = \frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}}{dt} \mathbf{e}_t + \mathbf{v} \frac{d\mathbf{e}_t}{d\theta} \frac{d\theta}{ds} \frac{ds}{dt} \]

\[ a_t = \frac{dv}{dt}, \quad a_n = \frac{v^2}{\rho} \]

\[ a = \frac{dv}{dt} \mathbf{e}_t + \frac{v^2}{\rho} \mathbf{e}_n \]

\[ \theta : \text{the angle between two tangential unit vectors} \]

\[ \frac{ds}{dt} = v \]

\[ \frac{d\theta}{ds} = \frac{1}{\rho} \rightarrow ds = \rho d\theta \]

\[ a = \frac{d\mathbf{v}}{dt} \mathbf{e}_t + \frac{v^2}{\rho} \mathbf{e}_n \]
Tangential and Normal Components

- Relations for tangential and normal acceleration also apply for a particle moving along a space curve.

- Plane containing tangential and normal unit vectors is called the osculating plane.

- Normal to the osculating plane is found from

\[
\vec{e}_b = \vec{e}_t \times \vec{e}_n
\]

\[
\vec{e}_n = \text{principal normal}
\]

\[
\vec{e}_b = \text{binormal}
\]

- Acceleration has no component along the binormal.
When the position of a particle moving in a plane is defined by its polar coordinates \( r \) and \( \theta \), it is convenient to use radial and transverse components directed, respectively, along the position vector \( \mathbf{r} \) of the particle and in the direction obtained by rotating \( \mathbf{r} \) through \( 90^o \) counterclockwise. Unit vectors \( \mathbf{e}_r \) and \( \mathbf{e}_\theta \) are attached to \( P \) and are directed in the radial and transverse directions. The velocity and acceleration of the particle in terms of radial and transverse components is

\[
\begin{align*}
\dot{e}_r &= \dot{\theta} \mathbf{e}_\theta \\
\dot{e}_\theta &= -\dot{\theta} \mathbf{e}_r \\
\nu &= \frac{d}{dt} (r \mathbf{e}_r) = \dot{r} \mathbf{e}_r + r \dot{\mathbf{e}}_r \\
\nu &= \dot{r} \mathbf{e}_r + r \dot{\mathbf{e}}_\theta \\
a &= (\ddot{r} - r \dot{\theta}^2) \mathbf{e}_r + (r \ddot{\theta} + 2r \dot{\theta}) \mathbf{e}_\theta \\
a_r &= \frac{d\nu_r}{dt}, \quad a_\theta = \frac{d\nu_\theta}{dt}
\end{align*}
\]

In 3-D

\[
\begin{align*}
\mathbf{r} &= R \mathbf{e}_R + z \mathbf{k} \\
\dot{r} &= \dot{z} \quad \text{and} \quad \ddot{z}
\end{align*}
\]

\[
\begin{align*}
V(\text{vector}) &= \dot{r} \mathbf{e}_r + r \dot{\theta} \mathbf{e}_\theta \\
a(\text{vector}) &= (\ddot{r} - r \dot{\theta}^2) \mathbf{e}_r + (r \ddot{\theta} + 2r \dot{\theta}) \mathbf{e}_\theta \\
\dot{\text{Dot}} &= \frac{d}{dt} \\
\dot{\text{Dotdot}} &= \frac{d^2}{dt^2}
\end{align*}
\]
Projectile Motion

airborne motion -- parabolic path of CM
constant horizontal velocity
constant vertical acceleration

Vertical velocity = 0
\[ a_x = 0 \quad a_y = -9.81 \text{ m/s}^2 \]
Loci of centers of rotation

\[ a_r = \ddot{r} - r \dot{\theta}^2, \quad a_\theta = r \ddot{\theta} + 2r \dot{\theta} \]
Three types of articulating surface motion in human joints.
Translational motion: (a) rectilinear; (b) curvilinear
Angular motion

General motion
Radial and Transverse Components

- When particle position is given in cylindrical coordinates, it is convenient to express the velocity and acceleration vectors using the unit vectors $\vec{e}_R$, $\vec{e}_\theta$, and $\vec{k}$.

- Position vector,

- Velocity vector,

- Acceleration vector,
A motorist is traveling on a curved section of highway at 60 mph. The motorist applies brakes causing a constant deceleration rate.

Knowing that after 8 s the speed has been reduced to 45 mph, determine the acceleration of the automobile immediately after the brakes are applied.
SOLUTION:

- Calculate tangential and normal components of acceleration.
- Determine acceleration magnitude and direction with respect to tangent to curve.

60 mph = 88 ft/s
45 mph = 66 ft/s

The components are dependent on the selection of coordinate system, but the resultant is the same in all coordinate systems.
If $V_B = 72/3.6$, then $a_n = 0.525 \text{ m/s}^2$ and $a = 0.984 \text{ m/s}^2$.
Rotation of the arm about O is defined by $\theta = 0.15t^2$ where $\theta$ is in radians and $t$ in seconds. Collar B slides along the arm such that $r = 0.9 - 0.12t^2$ where $r$ is in meters.

After the arm has rotated through $30^\circ$, determine (a) the total velocity of the collar, (b) the total acceleration of the collar, and (c) the relative acceleration of the collar with respect to the arm.

**SOLUTION:**
- Evaluate time $t$ for $\theta = 30^\circ$.
- Evaluate radial and angular positions, and first and second derivatives at time $t$.
- Calculate velocity and acceleration in cylindrical coordinates.
- Evaluate acceleration with respect to arm.
SOLUTION:
- Evaluate time $t$ for $\theta = 30^\circ$.

- Evaluate radial and angular positions, and first and second derivatives at time $t$. 
• Calculate velocity and acceleration.

\[ v = 0.524 \text{ m/s} \quad \beta = 31.0^\circ \]

\[ a = 0.531 \text{ m/s}^2 \quad \gamma = 42.6^\circ \]
- Calculate velocity and acceleration.

\[ \mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta \]

\[ \mathbf{a} = a_r \mathbf{e}_r + a_\theta \mathbf{e}_\theta \]

\[ a_t = a \cos(\gamma + \beta) = a \cos(72.6) = 0.159 \]

\[ a_n = a \sin(\gamma + \beta) = a \sin(72.6) = 0.507 \]

\[ \mathbf{v} = 0.524 \text{ m/s} \quad \beta = 31^\circ \]

\[ a = 0.531 \text{ m/s}^2 \quad \gamma = 42.6^\circ \]
The forces causing accelerations are applied to the system.

For example, there should be a force causing acceleration in radial and transverse directions.

In transverse direction this force bends the arm causing bending stress.

In radial direction there should be a rope, and tensile stress should be calculated or it starts motion.
- Evaluate acceleration with respect to arm.

Motion of collar with respect to arm is rectilinear and defined by coordinate $r$.

$$a_{B/OA} = \ddot{r} = -0.240 \text{ m/s}^2$$
The curves provided on the following pages represent instantaneous profiles of displacement ($D$), velocity ($V$), or acceleration ($A$) with respect to time ($T$). For each curve, enter the most appropriate time ($T_i$) that represents the following:

1. Peak positive displacement
2. Peak negative displacement
3. Instants of zero velocity
4. Instants of positive velocity
5. Instants of negative velocity
6. Peak positive velocity
7. Peak negative velocity
8. Instants of zero acceleration
9. Instants of positive acceleration
10. Instants of negative acceleration
11. Peak positive acceleration
12. Peak negative acceleration
1. Peak positive displacement
2. Peak negative displacement
3. Instants of zero velocity
4. Instants of positive velocity
5. Instants of negative velocity
6. Peak positive velocity
7. Peak negative velocity
8. Instants of zero acceleration
9. Instants of positive acceleration
10. Instants of negative acceleration
11. Peak positive acceleration
12. Peak negative acceleration
1. Peak positive displacement
2. Peak negative displacement
3. Instants of zero velocity
4. Instants of positive velocity
5. Instants of negative velocity
6. Peak positive velocity
7. Peak negative velocity
8. Instants of zero acceleration
9. Instants of positive acceleration
10. Instants of negative acceleration
11. Peak positive acceleration
12. Peak negative acceleration
1. Peak positive displacement
2. Peak negative displacement
3. Instants of zero velocity
4. Instants of positive velocity
5. Instants of negative velocity
6. Peak positive velocity
7. Peak negative velocity
8. Instants of zero acceleration
9. Instants of positive acceleration
10. Instants of negative acceleration
11. Peak positive acceleration
12. Peak negative acceleration
Given: \( x = 6t^2 - 8 + 40 \cos \pi t \quad x \sim \text{in.}, \ t \sim \text{s} \)

Find: \( x, \ s, \ \text{and} \ \alpha \) at \( t = 6 \text{s} \)

Have...
\( x = 6t^2 - 8 + 40 \cos \pi t \)
Then
\( s = \frac{dx}{dt} = 12t - 40\pi \sin \pi t \)
And
\( \alpha = \frac{ds}{dt} = 12 - 40\pi^2 \cos \pi t \)

At \( t = 6 \text{s}: \)
\( x_e = 6(6)^2 - 8 + 40 \cos 6\pi \quad \text{or} \quad x_e = 248 \text{ in.} \)
\( s_e = 12(6) - 40\pi \sin 6\pi \quad \text{or} \quad s_e = 72 \frac{\text{in.}}{\text{s}} \)
\( \alpha_e = 12 - 40\pi^2 \cos 6\pi \quad \text{or} \quad \alpha_e = -383 \frac{\text{in.}}{\text{s}^2} \)
11.19

**GIVEN:** \( a = k(1 - e^{-x}) \), \( k = \text{constant}; \)

When \( x = -2 \text{ m}, N = 6 \text{ m} \); when \( x = 0, N = 0 \)

**Find:**

(a) \( k \)

(b) \( N \) when \( x = -1 \text{ m} \)

---

(a) Have \( N \frac{dN}{dx} = a = k(1 - e^{-x}) \)

When \( x = -2 \text{ m}, N = 6 \text{ m} \):

\[ 
N \frac{dN}{dx} = \int_{x}^{N} k(1 - e^{-x}) \, dx 
\]

Or \( \frac{1}{2}(N^2 - 36) = k [x + e^{-x}]_{-2}^{x} \)

Or \( \frac{1}{2}N^2 = k(x + e^{-x} + 2 - e^{-2}) + 18 \)

When \( x = 0, N = 0 \):

\( 0 = k(1 + 2 - e^{2}) + 18 \)

Or \( k = 4.1011 \text{ m}^{2} \), \( k = 4.10 \frac{m^2}{s^2} \)

(b) When \( x = -1 \text{ m} \):

\( \frac{1}{2}N^2 = 4.1011(-1 + 1 + 2 - e^{2}) + 18 \)

Or \( N^2 = 2.43 \text{ m} \), \( N = 2.43 \text{ m} \)
**11.38**

**Given:** $a_{AB} = a_{CD} = 4.8 \, \text{m}^2$; $a_{BC}$ = constant; $t = 0, \Delta t = 0$, package at A; $a_d = 7.2 \, \text{m}$

**Find:**
(a) $d$
(b) $t_d$
(a) For A → B and C → D have: \[ N^2 = N_0^2 + 2a(x - x_0) \]
Then at B: \[ N_{BC}^2 = N_0^2 + 2(4.8 \frac{m}{s^2})(3 - 0)m \]
\[ = 28.8 \frac{m^2}{s^2} \quad (N_{BC} = 5.3666 \frac{m}{s}) \]
And at D: \[ N_D^2 = N_{BC}^2 + 2a_{CD}(x_D - x_C) \quad d = x_D - x_C \]
or \[ (7.2 \frac{m}{s})^2 = (28.8 \frac{m^2}{s^2}) + 2(4.8 \frac{m}{s^2})d \]
or \[ d = 2.40 \text{ m} \]

(b) For A → B and C → D have: \[ N = N_0 + at \]
Then at B: \[ 5.3666 \frac{m}{s} = 0 + (4.8 \frac{m}{s^2})t_{AB} \]
or \[ t_{AB} = 1.11804 \text{ s} \]
And at C → D: \[ 7.2 \frac{m}{s} = 5.3666 \frac{m}{s} + (4.8 \frac{m}{s^2})t_{CD} \]
or \[ t_{CD} = 0.38196 \text{ s} \]
Now for B → C have: \[ x_C = x_B + N_{BC}t_{BC} \]
or \[ 3 \text{ m} = (5.3666 \frac{m}{s})t_{BC} \]
or \[ t_{BC} = 0.55901 \text{ s} \]
Finally, \[ t_D = t_{AB} + t_{BC} + t_{CD} = (1.11804 + 0.55901 + 0.38196) \text{ s} \]
or \[ t_D = 2.065 \text{ s} \]
Given: Blocks A, B, and C and the pulley/cable system shown
From the diagram...

Cable 1: \(2y_A + 3y_B = \text{constant}\)

Then...

\(2x_A + 3x_B = 0 \quad (1)\)

And...

\(2x_A + 3x_B = 0 \quad (2)\)

Cable 2: \(y_B + 2y_C = \text{constant}\)

Then...

\(2x_B + 2x_C = 0 \quad (3)\)

And...

\(2x_B + 2x_C = 0 \quad (4)\)

---

11.49

**Given:** \(x_B = 24 \text{ in} \)

**Find:**

(a) \(x_A\)

(b) \(x_C\)

(c) \(x_D\)

(d) \(y_B\)

(a) Substituting into Eq. (1)...

\(2x_A + 3(24 \text{ in}) = 0\)

\(\text{or } x_A = -36 \text{ in}\)

(b) Substituting into Eq. (3)...

\((24 \text{ in}) + 2x_C = 0\)

\(\text{or } x_C = 12 \text{ in}\)

(c) From the diagram...

\(2y_A + y_B = \text{constant}\)

Then...

\(2x_A + x_B = 0\)

Substituting for \(x_A\)...

\(2(-36 \text{ in}) + x_B = 0\)

\(\text{or } x_B = 72 \text{ in}\)

(d) Have...

\(x_D/ \beta = x_D - x_B\)

\(= 72 \text{ in} - 24 \text{ in}\)

\(\text{or } x_D = 48 \text{ in}\)

---

\(3y_B - y_D = \text{constant}\)

\(3v_B - v_D = 0\)
\[2y_A + y_B + (y_B - y_E) = \text{constant} \rightarrow 2y_A + 2y_B - y_E = \text{constant}\]

\[y_B + y_E = \text{constant}\]
**11.97**

**GIVEN:** \( N_0 = 315 \text{ km} \); \( h = 80 \text{ m} \)

**Find:** \( d \)

---

**First Note:** \( N_0 = 315 \text{ km} \)

\[ \frac{1000}{3.6} \approx 87.5 \text{ m/s} \]

---

**Vertical Motion**

(Uniformly Accelerated Motion)

\[ y = y_0 + (u_{y0} t) - \frac{1}{2} g t^2 \]

At B, \( -80 \text{ m} = -\frac{1}{2}(9.81 \text{ m/s}^2) t^2 \)

\[ t_B = 4.038 \text{ s} \]

---

**Horizontal Motion** (Uniform)

\[ x = x_0 + (u_x) t \]

At B, \( d = (87.5 \text{ m/s})(4.038 \text{ s}) \)

\[ d = 353 \text{ m} \]
11.121

**GIVEN:** Constant velocities of trains A and B; at \( t=0 \), A is at the crossing; at \( t = 10 \) min, B is at the crossing.

**FIND:**
(a) \( \Delta B/A \)
(b) \( r_{B/A} \) at \( t = 3 \) min.
(a) Have \[ \sum S_B = \sum S_A + \sum S_{VA} \]

The graphical representation of this equation is then as shown.

\[ \sum S_A = 48 \text{ km \ h}^{-1} \]

\[ \sum S_B = 111.3 \text{ km \ h}^{-1} \]

Then.. \[ \sum S_{VA} = 66^2 + 48^2 - 2(66 \times 48) \cos 155^\circ \]

Or \[ \sum S_{VA} = 111.3 \text{ km \ h}^{-1} \]

And \[ \frac{48}{\sin \alpha} = \frac{111.3}{\sin 155^\circ} \]

Or \[ \alpha = 10.5^\circ \]

\[ \therefore \sum S_{VA} = 111.4 \text{ km \ h}^{-1} \angle 10.5^\circ \]

(b) First note that

At \( t = 3 \) min, \( A \) is \( (66 \text{ km \ h}^{-1}) \times \frac{3}{60} = 3.3 \text{ km \ west of the crossing} \).

At \( t = 3 \) min, \( B \) is \( (48 \text{ km \ h}^{-1}) \times \frac{3}{60} = 5.6 \text{ km \ southwest of the crossing} \).

Now- \[ \sum S_B = \sum S_A + \sum S_{VA} \]

Then at \( t = 3 \) min have..

\[ \sum S_A = 3.3 \text{ km} \]

\[ \sum S_{VA} = 3.3^2 + 5.6^2 - 2(3.3)(5.6) \cos 25^\circ \]

Or \[ \sum S_{VA} = 2.96 \text{ km} \]
Given: $R = 8.5 \text{ m}$

Find:
(a) $\Delta S_A$
(b) $\rho$ at the highest point of the trajectory.

$a_n = 9.81 \cos(40) = 7.52 \text{ m/s}^2$

$a_t = 9.81 \sin(40) = 6.30 \text{ m/s}^2$

$\Delta S_B = (\Delta S_A)_x$

$\Delta S_B = (\Delta S_A)_y = -g$

(a) Have...

$(Q_A)_n = \frac{\Delta S^2}{R}$

or $\Delta S^2 = (9.81 \cos 40^\circ)(8.5 \text{ m})$

or $\Delta S = 7.99 \text{ m}$

(b) Have...

$(Q_B)_n = \frac{\Delta S_B}{R_B}$

WHERE POINT $B$ IS THE HIGHEST POINT OF THE TRAJECTORY, SO THAT $N_B = (\Delta S_A)_x = \Delta S \cos 40^\circ$

THEN...

$R_B = \frac{(3.87 \text{ m/s}^2)^2 \cos 40^\circ}{9.81 \text{ m/s}^2}$

or $R_B = 3.82 \text{ m}$
GIVEN: A particle moves along the spiral shown.

Find: \( n^2 \) in terms of \( b \), \( \theta \), and \( \dot{\theta} \).

\[ n^2 = \frac{b^2}{\theta^2} \sqrt{4 + \theta^2} \]
Problem 11.184

A projectile enters a resisting medium at $x = 0$ with an initial velocity $v_0 = 274.32 \text{ m/s}$ and travels 10.16 cm before coming to rest. Assuming that the velocity of the projectile is defined by the relation $v = v_0 - kx$, where $v$ is expressed in m/s and $x$ is in m, determine (a) the initial acceleration of the projectile, (b) the time required for the projectile to penetrate 9.906 cm. into the resisting medium.
A projectile enters a resisting medium at $x = 0$ with an initial velocity $v_0 = 274.32$ m/s and travels 10.16 cm. before coming to rest. Assuming that the velocity of the projectile is defined by the relation $v = v_0 - kx$, where $v$ is expressed in m/s and $x$ is in m, determine (a) the initial acceleration of the projectile, (b) the time required for the projectile to penetrate 9.906 cm. into the resisting medium.

1. **Determine $a(x)$ for a given $v(x)$.** Substitute $v(x)$ in the formula

   $$a = v \frac{dv}{dx}$$

   Differentiate and obtain $a(x)$.

2. **Determine $t(x)$ for a given $v(x)$.** Substitute $v(x)$ in the formula

   $$v = \frac{dx}{dt}$$

   Rewrite as

   $$dt = \frac{dx}{v}$$

   Integrate and obtain $t(x)$. 
\[ v = v_0 - k x \]

Determine \( k \):

\[ 0 = 274.32 \text{ m/s} - k (0.1016 \text{ m}) \]

\[ k = 2700 \text{ s}^{-1} \]

Determine \( a(x) \) for a given \( v(x) \).

\[ a = v \frac{dv}{dx} \]

\[ a = (v_0 - k x)(-k) \]

(a) at \( x = 0 \) \( a = -k v_0 \)

\[ a = -(2700 \text{ s}^{-1})(274.32 \text{ m/s}) \]

\[ a = -740 664 \text{ m/s}^2 \]
Determine \( t(x) \) for a given \( v(x) \).

\[
v = \frac{dx}{dt}
\]

\[
v_o - k x = \frac{dx}{dt}
\]

\[
\int_0^t dt = \int_0^x \frac{dx}{\left( v_o - k x \right)} = \frac{-1}{k} \int_{v_o}^{v_o - k x} \frac{du}{u}
\]

\[
t = \frac{-1}{k} \ln \left( \frac{v_o - k x}{v_o} \right)
\]

At \( x = 0.09906 \) m

\[
t = \frac{-1}{2700} \ln \left[ 1 - \frac{2700 (0.09906)}{274.32} \right]
\]

\( t = 1.366 \times 10^{-3} \) s
Problem 11.186

In the position shown, collar $B$ moves to the left with a velocity of 150 mm/s. Determine (a) the velocity of collar $A$, (b) the velocity of portion $C$ of the cable, (c) the relative velocity of portion $C$ of the cable with respect to collar $B$. 
In the position shown, collar $B$ moves to the left with a velocity of 150 mm/s. Determine (a) the velocity of collar $A$, (b) the velocity of portion $C$ of the cable, (c) the relative velocity of portion $C$ of the cable with respect to collar $B$.

1. Dependent motion of two or more particles:

1a. **Draw a sketch of the system:** Select a coordinate system, indicating clearly a positive sense for each of the coordinate axes. The displacements, velocities, and accelerations have positive values in the direction of the coordinate axes.

1b. **Write the equation describing the constraint:** When particles are connected with a cable, its length which remains constant is the constraint.
In the position shown, collar $B$ moves to the left with a velocity of 150 mm/s. Determine (a) the velocity of collar $A$, (b) the velocity of portion $C$ of the cable, (c) the relative velocity of portion $C$ of the cable with respect to collar $B$.

1c. *Differentiate the equation describing the constraint:* This gives the corresponding relations among velocities and accelerations of the various particles.
Draw a sketch of the system.

Write the equation describing the constraint.

(a) The total length of the cable:

constant = 2x_A + x_B + (x_B - x_A)

Differentiate the equation describing the constraint.

0 = v_A + 2v_B

v_B = 150 mm/s

v_A = -2v_B

v_A = -300 mm/s  v_A = 300 mm/s
(b) The length of the cable from the right end to an arbitrary point on portion $C$ of the cable:

\[
\text{constant} = 2x_A + x_C
\]

\[
0 = 2v_A + v_C
\]

\[
v_C = -2v_A
\]

\[
v_C = 600 \text{ mm/s}
\]

(c) Relative velocity of portion $C$:

\[
v_C = v_B + v_{C/B}
\]

\[
600 = 150 + v_{C/B}
\]

\[
v_{C/B} = 450 \text{ mm/s}
\]
Problem 11.189

As observed from a ship moving due east at 8 km/h, the wind appears to blow from the south. After the ship has changed course, and as it is moving due north at 8 km/h, the wind appears to blow from the southwest. Assuming that the wind velocity is constant during the period of observation, determine the magnitude and direction of the true wind velocity.
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1. To solve relative motion problems:
   a. Write the vector equation that relates the velocities of two particles.

   \[ \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \]

   \( \mathbf{v}_B \) and \( \mathbf{v}_A \) are the velocities of particles \( B \) and \( A \) relative to a fixed frame, respectively. \( \mathbf{v}_{B/A} \) is the velocity of \( B \) relative to \( A \).
As observed from a ship moving due east at 8 km/h, the wind appears to blow from the south. After the ship has changed course, and as it is moving due north at 8 km/h, the wind appears to blow from the southwest. Assuming that the wind velocity is constant during the period of observation, determine the magnitude and direction of the true wind velocity.

b. Express all velocity vectors in terms of their rectangular components.

c. Scalar equations. Substitute the velocity vectors in $v_B = v_A + v_{B/A}$ and solve the two independent sets of scalar equations obtained.
Write the vector equation that relates the velocities of two particles.

\[ \mathbf{v}_w = \mathbf{v}_{s1} + \mathbf{v}_{w/s1} \]

Express all velocity vectors in terms of their rectangular components.

\[
\begin{align*}
\mathbf{v}_w &= v_{wx} \mathbf{i} + v_{wy} \mathbf{j} \\
\mathbf{v}_{s1} &= 8 \mathbf{i} \text{ km/h} \\
\mathbf{v}_{w/s1} &= v_{w/s1} \mathbf{j}
\end{align*}
\]

Scalar equations.

\[
\begin{align*}
v_{wx} &= 8 + 0 \\
v_{wy} &= 0 + v_{w/s1}
\end{align*}
\]

First observation

\[ v_{wx} = 8 \text{ km/h} \]
Problem 11.189 Solution

Write the vector equation that relates the velocities of two particles.

\[ \mathbf{v}_w = \mathbf{v}_{s2} + \mathbf{v}_{w/s2} \]

Express all velocity vectors in terms of their rectangular components.

\[ \mathbf{v}_w = v_{wx} \mathbf{i} + v_{wy} \mathbf{j} \quad \mathbf{v}_{s2} = 8 \mathbf{j} \quad \text{km/h} \]

\[ \mathbf{v}_{w/s2} = v_{w/s2} \cos 45^\circ \mathbf{i} + v_{w/s2} \sin 45^\circ \mathbf{j} \]

Scalar equations:

\[ v_{wx} = 0 + v_{w/s2} \cos 45^\circ \]

\[ v_{wy} = 8 + v_{w/s2} \sin 45^\circ \]

\[ v_{wy} = 16 \text{ km/h} \]

\[ \mathbf{v}_w = 8 \mathbf{i} + 16 \mathbf{j} \quad \text{km/h} \]

\[ \mathbf{v}_w = 17.89 \text{ km/h} \]

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Problem 11.190

The driver of an automobile decreases her speed at a constant rate from 45 to 30 mi/h over a distance of 750 ft along a curve of 1500-ft radius. Determine the magnitude of the total acceleration of the automobile after the automobile has traveled 500 ft along the curve.
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1. **Use tangential and normal components:** These components are used when the particle travels along a circular path. The unit vector $\mathbf{e}_t$ is tangent to the path (and thus aligned with the velocity) while the unit vector $\mathbf{e}_n$ is directed along the normal to the path and always points toward its center of curvature.

2. **Determine the tangential acceleration:** For a constant tangential acceleration, the acceleration can be determined from

$$\int_{v_o}^{v} v\, dv = a_t \int_{x_o}^{x} dx$$
The driver of an automobile decreases her speed at a constant rate from 45 to 30 mi/h over a distance of 750 ft along a curve of 1500-ft radius. Determine the magnitude of the total acceleration of the automobile after the automobile has traveled 500 ft along the curve.

3. **Determine the normal acceleration:** For known velocity and radius of curvature, the tangential acceleration is determined by

\[ a_n = \frac{v^2}{\rho} \]

4. **Determine the magnitude of the total acceleration:** The magnitude of the total acceleration is given by

\[ a = \sqrt{a_t^2 + a_n^2} \]
Determine the tangential acceleration.

\[
\int_{v_o}^v v \, dv = a_t \int_{x_o}^x dx
\]

\[
\int_{44}^{66} v \, dv = a_t \int_0^{750} dx
\]

\[
\frac{1}{2} (44^2 - 66^2) = a_t (750 - 0)
\]

\[
a_t = -1.613 \text{ ft/s}^2
\]
Problem 11.190 Solution

The speed after the automobile traveled 500 ft.

\[ \int_{v_0}^{v} v \, dv = a_t \int_{x_0}^{x} dx \]

\[ \int_{66}^{v_1} v \, dv = -1.613 \int_{0}^{500} dx \]

\[ \frac{1}{2} (v_1^2 - 66^2) = -1.613 (500 - 0) \]

\[ v_1 = 52.4 \text{ ft/s} \]
Problem 11.190 Solution

Determine the normal acceleration.

\[ a_n = \frac{v^2}{\rho} \]

\[ a_n = \frac{(52.4 \text{ ft/s})^2}{1500 \text{ ft}} = 1.828 \text{ ft/s}^2 \]

Determine the magnitude of the total acceleration.

\[ a = \sqrt{a_t^2 + a_n^2} \]

\[ a = \sqrt{(-1.613 \text{ ft/s}^2)^2 + (1.828 \text{ ft/s}^2)^2} \]

\[ a = 2.44 \text{ ft/s}^2 \]
Problem 11.191

Knowing that the conveyor belt moves at the constant speed $v_0 = 7.315$ m/s, determine the angle $\alpha$ for which the sand is deposited on the stockpile at $B$. 

\[ \alpha \]

4.572 m

\[ \alpha \] 

7.62 m
Problem 11.191

Solving Problems on Your Own

Knowing that the conveyor belt moves at the constant speed $v_0 = 24 \text{ ft/s}$, determine the angle $\alpha$ for which the sand is deposited on the stockpile at $B$.

1. Analyzing the motion of a projectile: Consider the vertical and the horizontal motion separately.

2. Consider the horizontal motion: When the resistance of the air can be neglected, the horizontal component of the velocity remains constant (uniform motion). The distance, velocity, and time are related by

$$x = x_0 + (v_x)_0 \ t$$
Problem 11.191
Solving Problems on Your Own

Knowing that the conveyor belt moves at the constant speed \( v_0 = 24 \text{ ft/s} \), determine the angle \( \alpha \) for which the sand is deposited on the stockpile at \( B \).

3. Consider the vertical motion: When the resistance of the air can be neglected, the vertical component of the acceleration is constant (uniformly accelerated motion). The distance, velocity, acceleration and time are related by

\[
y = y_0 + (v_y)_0 t - \frac{1}{2} g t^2
\]
Consider the horizontal motion.

\[ x = x_0 + (v_x)_0 \, t \]
\[ x = +7.62 \, \text{m} \]
\[ x_0 = 0 \]
\[ (v_x)_0 = v_0 \cos \alpha \]
\[ (v_x)_0 = (7.315 \, \text{m/s}) \cos \alpha \]

Consider the vertical motion

\[ y = y_0 + (v_y)_0 \, t - \frac{1}{2} \, g \, t^2 \]
\[ y = -4.572 \, \text{m} \]
\[ (v_y)_0 = v_0 \sin \alpha \]
\[ (v_y)_0 = (7.315 \, \text{m/s}) \sin \alpha \]
\[ y_0 = 0 \]

\[ (-4.572 \, \text{m}) = 0 + (7.315 \, \text{m/s}) \sin \alpha \, t - \frac{1}{2} \, (9.81 \, \text{m/s}^2) \, t^2 \]

Two equations with two unknowns, \( \alpha \) and \( t \).
Problem 11.191 Solution

\[ 25 = 0 + 24 \cos \alpha t \]
\[ -15 = 0 + 24 \sin \alpha t - \frac{1}{2} 32.2 \, t^2 \]

Solve the equations for \( \alpha \) and \( t \).

eliminate \( \alpha \): \[ \sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \left( \frac{25}{24} t \right)^2} \]

\[-15 = 24 \, t \sqrt{1 - \left( \frac{25}{24} t \right)^2} - 16.1 \, t^2 \]
\[ 250.21 \, t^4 - 1059 \, t^2 + 850 = 0 \]

two solutions:
\[ t = 1.048 \, \text{s} , \quad \alpha = 6.09^\circ \]
\[ t = 1.729 \, \text{s} , \quad \alpha = 52.9^\circ \]

\( \alpha = 6.09^\circ \) or \( 52.9^\circ \)
\[
25 = 0 + 24 \cos \alpha t
\]

\[
-15 = 0 + 24 \sin \alpha t - \frac{1}{2} 32.2 t^2
\]

\[\cos \alpha = \frac{25}{24t}\]

\[24t \quad \alpha \quad 25 \quad \sqrt{(24t)^2 - 25^2}\]

\[
-15 = 24 \frac{\sqrt{(24t)^2 - 25^2}}{24t} t - \frac{1}{2} gt^2
\]

\[
-15 = \sqrt{(24t)^2 - 25^2} - \frac{1}{2} gt^2
\]

\[
-15 + \frac{1}{2} gt^2 = \sqrt{(24t)^2 - 25^2}
\]

\[
\left(-15 + \frac{1}{2} gt^2\right)^2 = (24t)^2 - 25^2
\]
\[
25 = 0 + 24 \cos \alpha t \\
\]
\[
-15 = 0 + 24 \sin \alpha t - \frac{1}{2} 32.2 \ t^2 \\
\]
\[
\frac{25}{24} = \cos \alpha t \\
\]
\[
\frac{-15}{24} + \frac{1}{2 \times 24} \gt^2 = \sin \alpha t \\
\]
\[
\left( \frac{25}{24} \right)^2 = (\cos \alpha t)^2 \\
\]
\[
\left( \frac{-15}{24} + \frac{1}{2 \times 24} \gt^2 \right)^2 = (\sin \alpha t)^2 \\
\]
\[
\left( \frac{25}{24} \right)^2 + \left( \frac{-15}{24} + \frac{1}{2 \times 24} \gt^2 \right)^2 = t^2 \\
\]
\[25 = 0 + 24 \cos \alpha t\]
\[-15 = 0 + 24 \sin \alpha t - \frac{1}{2} 32.2 t^2\]

\[t = \frac{25}{24 \cos \alpha}\]

\[-15 = 24 \sin \alpha \frac{25}{24 \cos \alpha} - \frac{1}{2} g \left( \frac{25}{24 \cos \alpha} \right)^2\]

\[-15 = 25 \tan \alpha - \frac{1}{2} g \left( \frac{25}{24} \sec \alpha \right)^2\]

\[-15 = 25 \tan \alpha - \frac{1}{2} g \left( \frac{25}{24} \right)^2 \sec^2 \alpha\]

\[-15 = 25 \tan \alpha - \frac{1}{2} g \left( \frac{25}{24} \right)^2 (1 + \tan^2 \alpha)\]

\[-\frac{1}{2} g \left( \frac{25}{24} \right)^2 \tan^2 \alpha + 25 \tan \alpha - \frac{1}{2} g \left( \frac{25}{24} \right)^2 + 15 = 0\]
clear all;clf;clc
syms t
x=30*t^2-120*t;
vx=diff(x);
ax=diff(vx);
y=120*t^2-40*t^3;
vy=diff(y);
ay=diff(vy);
figure(1);
%subplot(3,1,1); ezplot(x,[0,2])
%subplot(3,1,2);ezplot(vx,[0,2])
%subplot(3,1,3);ezplot(ax,[0,2])
figure(2);
%subplot(3,1,1); ezplot(y,[0,2])
%subplot(3,1,2);ezplot(vy,[0,2])
%subplot(3,1,3);ezplot(ay,[0,2])
figure(1)
subplot(3,2,1); ezplot(x,[0,2])
subplot(3,2,3);ezplot(vx,[0,2])
subplot(3,2,5);ezplot(ax,[0,2])
subplot(3,2,2); ezplot(y,[0,2])
subplot(3,2,4);ezplot(vy,[0,2])
subplot(3,2,6);ezplot(ay,[0,2])