Chapter 3  KINETICS OF PARTICLES: ENERGY AND MOMENTUM
METHODS

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Introduction

• Previously, problems dealing with the motion of particles were solved through the fundamental equation of motion, $\vec{F} = m\vec{a}$. Current chapter introduces two additional methods of analysis.

• *Method of work and energy:* directly relates force, mass, velocity and displacement.

• *Method of impulse and momentum:* directly relates force, mass, velocity, and time.

\[
F = ma \Rightarrow \begin{cases} 
\text{integrate with respect to time} & I = \int m\,ad\,t \\
\text{integrate with respect to space} & U = \int m\,a\,d\,s \end{cases}
\]
Work of a Force

- Differential vector $d\vec{r}$ is the *particle displacement*.

- Work of the force is

$$dU = \vec{F} \cdot d\vec{r}$$

$$= F \, ds \, \cos \alpha$$

$$= F_x dx + F_y dy + F_z dz$$

- Work is a scalar quantity, i.e., it has magnitude and sign but not direction.

- Dimensions of work are length $\times$ force. Units are

$1 \text{ J} \equiv \text{N m}$

$$1 \text{ ft} \cdot \text{lb} = 1.356 \text{ J}$$
Consider a force $\mathbf{F}$ acting on a particle $A$. The work of $\mathbf{F}$ corresponding to the small displacement $\mathbf{d}r$ is defined as

$$dU = \mathbf{F} \cdot \mathbf{d}r$$

Recalling the definition of scalar product of two vectors,

$$dU = F ds \cos \alpha$$

where $\alpha$ is the angle between $\mathbf{F}$ and $\mathbf{d}r$. 
The work of \( \mathbf{F} \) during a finite displacement from \( A_1 \) to \( A_2 \), denoted by \( U_{1\rightarrow2} \), is obtained by integrating along the path described by the particle.

\[
U_{1\rightarrow2} = \int_{A_1}^{A_2} \mathbf{F} \cdot d\mathbf{r}
\]

Work is a scalar quantity.

N.m (Joule), ft.lb or in.lb

1 ft.lb = (0.3048 m)(4.448 N) = 1.356 J

\( \alpha \) is acute, then \( U > 0 \)

\( \alpha \) is obtuse, then \( U < 0 \)

\( \alpha = 0 \), then \( U = F \cdot ds \)

\( \alpha = 180^\circ \), then \( U = -F \cdot ds \)

\( \alpha = 90^\circ \), then \( U = 0 \)

If \( F \) is constant (in magnitude and direction)

\( U_{1\rightarrow2} = \mathbf{F} \cos \alpha \Delta x \)

For a force defined by its rectangular components, we write

\[
U_{1\rightarrow2} = \int_{A_1}^{A_2} (F_x \, dx + F_y \, dy + F_z \, dz)
\]
\( \alpha \) is acute, then \( U > 0 \)
\( \alpha \) is obtuse, then \( U < 0 \)
\( \alpha = 0 \), then \( U = F \cdot ds \)
\( \alpha = 180^\circ \), then \( U = -F \cdot ds \)
\( \alpha = 90^\circ \), then \( U = 0 \)
If $F$ and $d_{\parallel}$ point in the same direction, the work done is positive.

If $F$ and $d_{\parallel}$ point in the opposite direction, the work done is negative.
• Work of a constant force in rectilinear motion,

$$U_{1\rightarrow 2} = \mathbf{F} \cos \alpha \Delta x$$

When \( \mathbf{F} \) is not parallel to \( \mathbf{d} \), then we must take the component of \( \mathbf{F} \) which is parallel to \( \mathbf{d} \).

\[
\begin{align*}
F_h &= F \cos \theta \\
F_v &= F \sin \theta \\
W_k &= F_h d = F (\cos \theta) d
\end{align*}
\]

If \( F = 100 \text{ N} \) and \( \theta = 30 \text{ degrees} \),
Compute \( F_v \) and \( F_h \)
If \( d = 5 \text{ m} \), compute \( W \)
Example:
If you pull on a wagon with a force of 100 N at an angle of 30 degrees w.r.t. the horizontal and you pull over a distance of 5 m, how much work do you do on the wagon?
Know: $F = 100 \text{ N}; \, d = 5 \text{ m}; \, \theta = 30^\circ$

Need: $W_k$

Use: $W_k = (F \cdot d)_{\text{parallel}} = F \cdot \cos \theta \cdot d$

$W_k = 100 \text{ N} \cdot \cos 30^\circ \cdot 5 \text{ m} = 433 \text{ N} \cdot \text{m}$
The work of the weight \( W \) of a body as its center of gravity moves from an elevation \( y_1 \) to \( y_2 \) is obtained by setting \( F_x = F_z = 0 \) and \( F_y = -W \).

\[
dU = -W\,dy
\]

\[
U_{1\to2} = -\int_{y_1}^{y_2} W\,dy = -W(y_2 - y_1) = W(y_1 - y_2)
\]

or

\[
U_{1\to2} = -W(y_2 - y_1) = -W\Delta y
\]

- Work of the weight is positive when \( \Delta y < 0 \), i.e., when the weight moves down.

The work is **negative when the elevation increases**, and **positive when the elevation decreases**.
If a system does positive work (the force exerted by the system is in the direction of the motion), then its energy decreases.

If a system does negative work (the force exerted by the system is in the opposite direction of the motion), then its energy increases.

If a system does negative work, actually it does not work, an external force does work on the system, that’s why the energy increases.

\[ Q = PdV + VdP \]

\[ U = Fh \]

\[ U = P_Ah \]

\[ U = (P_{atm}A + W)h \]
Work of the Force Exerted by a Spring

The work of the force $F$ exerted by a spring on a body $A$ during a finite displacement of the body from $A_1 (x = x_1)$ to $A_2 (x = x_2)$ is obtained by writing

$$dU = -Fdx = -kx\,dx$$

$$U_{1\rightarrow2} = -\int_{x_1}^{x_2} kx\,dx = \frac{1}{2} kx_1^2 - \frac{1}{2} kx_2^2$$

The work is positive when the spring is returning to its undeformed position, when $x_2 < x_1$.

- Work of the force exerted by the spring is equal to negative of area under curve of $F$ plotted against $x$, $U_{1\rightarrow2} = -\frac{1}{2} (F_1 + F_2) \Delta x$
A 100 N external force is applied to a spring with \( k=1000 \) N/m.

\[
x = \frac{F}{k} = \frac{100}{1000} = 0.1 \text{ m compression or extension amount}
\]

Use it to find the energies

\[
V_{e1} = 0
\]

\[
V_{e2} = \frac{1}{2} kx^2 = \frac{1}{2} (1000)(0.1^2) = 5 \text{ J energy is stored in the spring}
\]

\[
\Delta V_e = -5 \text{ J work is done by the spring}
\]

or

\[
U_{1\rightarrow2} = -\frac{1}{2} F \Delta x = -\frac{1}{2} (100)(0.1) = -5 \text{ J work is done by the spring}
\]
\[ U_{1 \rightarrow 2} = \frac{1}{2} \theta + F \Delta x = 5 \text{ J work is done by the external force} \]

or

\[ U_{1 \rightarrow 2} = -\Delta V_e = \frac{1}{2} kx^2 = 5 \text{ J work is done by the external force} \]
If you are compressing or stretching an undeformed spring, the spring force always does negative work (gaining energy).

If the spring is initially stretched or compressed and you release it, then the spring force does positive work. (losing energy)
The work of the gravitational force $\mathbf{F}$ exerted by a particle of mass $M$ located at $O$ on a particle of mass $m$ as the latter moves from $A_1$ to $A_2$ is obtained from

$$dU = -Fdr = -G \frac{Mm}{r^2} dr$$

$$U_{1\to2} = - \int_{r_1}^{r_2} G \frac{Mm}{r^2} dr = G \frac{Mm}{r_2} - G \frac{Mm}{r_1}$$

$$= GMm \left( \frac{1}{r_2} - \frac{1}{r_1} \right)$$
Work of a Force

Forces which *do not* do work \((ds = 0 \text{ or } \cos \alpha = 0)\):

- reaction at frictionless pin supporting rotating body,
- reaction at frictionless surface when body in contact moves along surface,
- reaction at a roller moving along its track, and
- weight of a body when its center of gravity moves horizontally.
Team A
−1,000,000 N

d = 1 m

Team B
+1,000,001 N

\[ \Sigma F = -1,000,000 \text{ N} + 1,000,001 \text{ N} = 1 \text{ N} \]

\[ W_k = \Sigma F \cdot d = 1 \text{ N} \cdot 1 \text{ m} = 1 \text{ Nm} \]
Energy = the capacity to do work (scalar)

Types of energy: mechanical, chemical, heat, sound, light, etc. We are most interested in mechanical energy, but in bioengineering, all of them should be considered.
Particle Kinetic Energy: Principle of Work & Energy

- Consider a particle of mass \( m \) acted upon by force \( \vec{F} \)

\[
F_t = ma_t = m \frac{dv}{dt} = m \frac{dv}{ds} \frac{ds}{dt} = mv \frac{dv}{ds}
\]

\[
F_t \, ds = mv \, dv
\]

- Integrating from \( A_1 \) to \( A_2 \),

\[
\int_{s_1}^{s_2} F_t \, ds = m \int_{v_1}^{v_2} v \, dv = \frac{1}{2} mv_2^2 - \frac{1}{2} mv_1^2
\]

\[
U_{1 \rightarrow 2} = T_2 - T_1 \quad T = \frac{1}{2} mv^2 = \text{kinetic energy}
\]

- The work of the force \( \vec{F} \) is equal to the change in kinetic energy of the particle.

- Units of work and kinetic energy are the same:

\[
T = \frac{1}{2} mv^2 = \text{kg} \left( \frac{m}{s^2} \right)^2 = \left( \text{kg} \, \frac{m}{s^2} \right) m = \text{N} \cdot \text{m} = \text{J}
\]
Mechanical Energy

- Kinetic Energy \( (T) \) - energy due to **motion**

\[
T = \frac{1}{2} mv^2 = \text{kg} \left( \frac{\text{m}}{\text{s}} \right)^2 = \left( \text{kg} \frac{\text{m}}{\text{s}^2} \right) m = \text{N} \cdot \text{m} = \text{J}
\]

e.g. a diver (mass = 70 kg) hits the water after a dive from the 10 m tower with a velocity of 14 m/s. How much \( T (\text{KE}) \) does she possess?

\[
T = \frac{1}{2} mv^2 = \frac{1}{2} \cdot 70 \cdot 14^2 = 6860 \text{ J}
\]
The kinetic energy of a particle of mass $m$ moving with a velocity $v$ is defined as the scalar quantity

$$ T = \frac{1}{2} \ m v^2 $$

J or ft.lb

From Newton’s second law, the principle of work and energy is derived. This principle states that the kinetic energy of a particle at $A_2$ can be obtained by adding to its kinetic energy at $A_1$ the work done during the displacement from $A_1$ to $A_2$ by the force $\mathbf{F}$ exerted on the particle:

$$ T_1 + U_{1\rightarrow 2} = T_2 $$
• Work of friction forces is always negative.
• Reaction forces or normal forces along frictionless surfaces, weight when it moves horizontally do not work
• Static friction force does not work
• Kinetic friction force does work
• Constant velocity motion

\[ W = mg \]

No slip

Work of deformation
• Deceleration without gear box

\[ W = mg \]

\[ N \]

\[ F_s = \mu_s N \]

No slip

Work of deformation
- Brakes are applied

\[ W = mg \]

\[ F_k = \mu_k N \quad \text{(if it skids)} \]

\[ F_s = \mu_s N \quad \text{(if it does not skid, e.g. ABS)} \]

- Ayrıca kaslar kısalırken iş yapar, uzarken üzerinde iş yapılır.
- Also during extension of muscles, some work is done.

- The weight of the car was not doing any work in horizontal direction.
- The weight of a person does work when he/she is moving in the horizontal direction. The center of gravity of that person is moving up and down continually.

$$1000 \times 70 \times 9.81 \times 0.04 = 70 \times 9.81 \times 40$$
The hyperbolic force–velocity relation, which is valid on all possible levels (from a single muscle fiber level to the global human musculo–skeletal apparatus) shows that a muscle can generate its maximum force in isometric conditions (without motion), and that it can contract with its maximum velocity (i.e., speed) in isotonic conditions (without external load). The area under the force–velocity hyperbole corresponds to the mechanical power (dimensionally force $\times$ velocity $[N \cdot ms^{-1}]$, exactly defined as $\int v^2 Fdv$), of a single muscle contraction.
Therefore, it tells us that we can have either force (strength) or velocity (speed) in our muscular contraction, but not both. If we want both of them, and it is so–called power, we must shift the curve up–and–right, which corresponds to the development of the muscle power. The asymptotes \textit{a and b of the curve, having respectively dimensions of force \([N]\) and velocity \([m/s]\)}, correspond to \textit{the amount of energy dissipated} during the contraction and velocity of chemo-mechanical processes involved. Hill’s mechanical as well as thermodynamic equation has the form: \((F +a)(v +b) = (F_o+a)b\). \textit{This is essential muscular characteristic for start} acceleration in sprint running (see the next paragraph), and can be indirectly measured by varying loadings and performing maximal speed movements.
Work-Energy Relationship

The work done by the net force acting on a body is equal to the change in the body’s kinetic energy.

\[
(\sum F \cdot d_{\text{parallel}}) = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2
\]

\[
\sum W_k = \Delta KE
\]

This relationship is true as long as there is no change in vertical position.
if $KE_i = 0$ and $KE_f = 0$ then $\Delta KE = 0$; therefore the work done by the person (GRF$_v$) is completely offset by the negative work done by gravity (W).

Overall -- no work was done because there was no change in KE overall

What about muscles? Is that true?
Consider the **work done by a single force. In this example consider** $F_v$

\[ F_v \cdot d_v = \Delta KE + \Delta PE (+\Delta SE) \]
\[ \Delta KE = 0 \]
\[ \Delta PE = -mg\Delta h = -100\text{kg}(-9.8\text{m/s}^2)2\text{m} = 1960\text{J} \]

This is numerically opposite to the work done by gravity.

$F_v$ did +1960 J of work
$W$ did −1960 J of work
Applications of the Principle of Work and Energy

- Wish to determine velocity of pendulum bob at $A_2$. Consider work & kinetic energy.
- Force $\vec{P}$ acts normal to path and does no work.

\[
T_1 + U_{1\rightarrow 2} = T_2
\]

\[
0 + Wl = \frac{1}{2} \frac{W}{g} v_2^2
\]

\[
v_2 = \sqrt{2gl}
\]

- Velocity found without determining expression for acceleration and integrating.
- All quantities are scalars and can be added directly.
- Forces which do no work are eliminated from the problem.
Wish to determine velocity of pendulum bob at $A_2$. Consider work & kinetic energy.

$$ds = -ld\theta$$

$\theta$ from 90 to 0 degrees
s from 0 to quarter circular arc
So $\theta$ decreases $s$ increases

$$T_1 + \int Fds = T_2$$

$$T_1 + \int_{0}^{90} mg\sin\theta l\quad d\theta = T_2$$

$$T_1 - mgl \int_{90}^{0} \sin \theta d\theta = T_2$$

$$T_1 - mgl \int_{0}^{90} \sin \theta d\theta = T_2$$

$$T_1 - mgl \quad -\cos \theta \quad _{90}^{0} = T_2$$

$$T_1 + mgl \quad 1 - \cos 90 \quad = T_2$$

$$T_1 + mgl \quad 1 = T_2$$
\[ \Sigma F_t = ma_t \]

\[ mg \sin \theta = ma_t, \]
\[ g \sin \theta = a_t \]

\[ \int_0^s g \sin \theta ds = \int_0^s a_t ds = \int_0^v v dv \]

\[ \int_0^{90} g \sin \theta l -d\theta = \int_{90}^{90} a_t l -d\theta \]

\[ mgsin\theta \]
• Principle of work and energy cannot be applied to directly determine the acceleration of the pendulum bob.

• Calculating the tension in the cord requires supplementing the method of work and energy with an application of Newton’s second law.

• As the bob passes through $A_2$,

\[ \sum F_n = m a_n \]

\[ P - W = \frac{W v_2^2}{g l} \]

\[ P = W + \frac{W 2 gl}{g l} = 3W \]

\[ v_2 = \sqrt{2gl} \]
The **mechanical work** done by children during walking at different speeds can be divided into different parts: (1) the external work necessary to sustain the displacement of the center of mass of the body relative to the surroundings, (2) the internal work done to accelerate the body segments relative to the center of mass of the body and (3) the internal work done during the double contact phase of walking by the back leg which generates energy that will be absorbed by the front leg (see picture). This last part is not counted in the 'classic' measurements of the positive muscular work done during walking. Using force platforms, one can study the effect of speed and age (size) on the internal work done by one leg against the other. One can also measure the total mechanical work done during walking at different speeds in adults and in 3-12 years old children.
The *bouncing mechanism* of running in children

the mechanical work done to move the center of mass of the body during running at various speeds in children of different age

Why do we need work to do this?
- the **pendular mechanism** of walking

- the **bouncing mechanism** of running trotting and hopping

In the figure here above, the upper curve ($E_{kf}$) represent the kinetic energy changes due to the forward velocity of the center of mass of the body ($CG$). The continuous line in the middle trace ($E_p+E_{kv}$) represents the sum of the kinetic ($E_{kv}$) et potential energy ($E_p$, dotted line) due to the vertical movements of the $CG$. The lower curve ($E_{cg}$) represents the energy changes of the $CG$. $E_{cg}$ is the sum of the other curves ($E_{cg} = E_{kf}+E_p+E_{kv}$). In walking, $E_{kf}$ and $E_p+E_{kv}$ are in opposition of phase i.e. kinetic energy is transformed into potential energy and vice-versa. Due to this pendular mechanism, the variations of $E_{cg}$ are reduced and the muscular work done to move the center of mass of the body is decreased. In running, $E_{kf}$ and $E_p+E_{kv}$ are in phase, suggesting a bouncing mechanism. The movements of the center of mass of the body can be compared to those of a spring-mass system. During the aerial phase, $E_{cg}$ is constant (the air friction is negligible). At the beginning of the contact phase, the center of mass of the body looses speed and height. The energy lost during this negative work phase is stored in the muscles and tendons and is recovered during the phase of positive work production to elevate and accelerate the center of mass of the body.
Mechanical Energy

- (Gravitational) Potential Energy, $U$ (P.E.)
  - energy due to the **change of position in** gravitational field
  - $PE = -mgh$
  
  $h$ = height of something above some reference line
  
  $m$ = **mass**
  
  $g$ = *acceleration due to gravity* (~$9.8 \text{ m/s}^2$)
Potential Energy

- Work of the force of gravity $\vec{W}$,
  \[ U_{1 \rightarrow 2} = W y_1 - W y_2 \]

- Work is independent of path followed; depends only on the initial and final values of $W_y$.
  \[ V_g = W_y \]
  \[ = \text{potential energy of the body with respect to force of gravity.} \]
  \[ U_{1 \rightarrow 2} = \int_{y_1}^{y_2} \mathbf{F}_g \cdot d\mathbf{r} \]

- Choice of datum from which the elevation $y$ is measured is arbitrary.

- Units of work and potential energy are the same:
  \[ V_g = W_y = N \cdot m = J \]
\[\text{m} = 1 \text{ kg}\]
\[v_i = 6.3 \text{ m/s}\]
\[ \Delta PE > 0 \]
\[ \Delta PE = -m(-g)(+h) \]
\[ \Delta PE = +mgh \]

\[ \Delta PE < 0 \]
\[ \Delta PE = -m(-g)(-h) \]
\[ \Delta PE = -mgh \]

Units: $kg \cdot m^2 /s^2 = J$

The negative sign in the PE equation is necessary to account for the direction of gravity

(PE > 0 when h > 0 and PE < 0 when h < 0)
Potential Energy:
Note: in the absence of air resistance and other resistive forces -- PE can be completely converted to KE by the work done by gravity on the way down.

e.g. a diver on top of a 10 m tower has a positive ΔPE compared to water level

\[ \Delta PE = -mgh = -60 \text{ kg} \cdot (-9.8 \text{ m/s}^2) \cdot (+10 \text{ m}) = 6860 \text{ J} \]

NOTE: This value is identical to that found in the kinetic energy example.
• Previous expression for potential energy of a body with respect to gravity is only valid when the weight of the body can be assumed constant.

• For a space vehicle, the variation of the force of gravity with distance from the center of the earth should be considered.

• Work of a gravitational force,

\[
U_{1\rightarrow 2} = \frac{GMm}{r_2} - \frac{GMm}{r_1}
\]

• Potential energy \( V_g \) when the variation in the force of gravity can not be neglected,

\[
V_g = -\frac{GMm}{r} = -\frac{WR^2}{r}
\]
Mechanical Energy
• Strain or elastic energy (SE)
  – energy due to deformation
  – this type of energy arises in compressed springs, squashed balls ready to rebound, stretched tendons inside the body, and other deformable structures

\[ SE = \frac{1}{2} kx^2 \]

\( x \) : amount of deformation
\( k \) : stiffness of spring or other structure
• Work of the force exerted by a spring depends only on the initial and final deflections of the spring,

\[ U_{1\to2} = \frac{1}{2} kx_1^2 - \frac{1}{2} kx_2^2 \]

• The potential energy of the body with respect to the elastic force,

\[ V_e = \frac{1}{2} kx^2 \]

\[ U_{1\to2} = V_e_1 - V_e_2 \]

• Note that the preceding expression for \( V_e \) is valid only if the deflection of the spring is measured from its undeformed position.
Anisotropic Behavior of the Bone

- **anisotropy**: the property of a material which exhibits different mechanical properties when loaded in different directions.

- Stiffness with respect to tension is maximal for axial loads and minimal for perpendicular loads.

- For ultimate stress of cortical bone: compression > tension > shear.
Strain Rate Dependency
The stiffness of a bone changes with the rate of loading

- when loads are applied at higher rate within the physiological limit, the bone
  - becomes stiffer
  - sustains a higher load to failure
  - stores more energy before failure
- when a bone fractures, the stored energy is released.
  - single bone crack for a low-energy fracture
  - comminuted fracture of bone for a higher-energy fracture
  - severe destruction of bone before failure
Immobilization

Effect of Mobility

- Normal activity
- Immobilization

Degeneration

Effect of Degeneration

- Young
- Aged
Mechanical Properties of the Collagen Fibers

Structure
- the most abundant protein in the body
- to resist tensile stress
- tropocollagen: 3 procollagen polypeptide chains (α chains) coiled about each other into a right-handed triple helixes

Types
- Type I
  - found in bone, tendon, ligament, and skin
- Type II
  - found in articular cartilage, nasal septum, and sternal cartilage

Tensile Strength

![Stress-Strain Curve of Collagen Fiber](image-url)
Hysteresis

- Energy stored in a viscoelastic material when a load is given and then relaxed.

Example of heat dissipation
One can not regain the energy transferred

-aged heel pad poor ability to absorb the shock
Mechanical Properties of the Skeletal Muscle

Length-Tension Relationship

- The tension that a muscle generates varies with its length
- Found when a muscle under isometric contraction and for maximum activation of the muscle
- In a single muscle fiber,
  - Peak force is noted at normal resting length.
  - A bell-shaped length-tension curve

In a muscle, force generation capacity increases when the muscle is slightly stretched because of the effect of both active and passive components.
sliding filament theory and length/tension curve
- theory proposed by biophysicist Jean Hanson (1919-73) and physiologist Hugh Esmor Huxley (1924- ) in 1954
- states that during contraction thin filaments slide past thick filaments with no change in the length of either type of filament
- force for producing sliding of thin filaments is generated by the cross-bridges (formed by myosin heads)
- theory predicts that force output will be proportional to the degree of overlap between thick and thin filaments or, more specifically, the number of cross-bridges formed

biomechanical implications of the sliding filament theory:
1. for maximum force output (total tension) muscle should be positioned below its optimal length so that work (either positive or negative) will occur over peak of length/tension curve
2. muscles which produce the same action across a joint are typically arranged such that their optimal lengths occur at different joint positions thus permitting a nearly constant level of force output at all joint positions
The force-generating capacity of a sarcomere depends strongly on the degree of overlap of myosin and actin filaments.
Stress-strain behavior of typical materials:

a) non-biological material
b) Viscoelastic structure (tendon)
Workloop Methods

Length
Stim
Force
Time →

Work Out − Work In = Net Work
Stress-strain (or load-extension) behavior of ligament loaded in tension
1) Toe region
2) Almost linear region, stiffness nearly constant
3) Failure region
The three regions of the active component of the Length–Tension Relationship. Differences in the work \( (W = F \cdot d) \) the muscle can do in the ascending limb \( (W_A) \) versus the plateau region \( (W_P) \) are illustrated. Work can be visualized as the area under a force–displacement graph.
Stress-strain curves measure fracture resistance, or energy to failure. In our study of bone from the same three individuals featured in Table, our assembly could also measure fracture resistance and plot it for comparison. Fracturing behaviors we observed in videos of the fracturing bone correlated with these mechanical testing results; the circles in each line correspond to a fracture behavior.

<table>
<thead>
<tr>
<th>Donor</th>
<th>E [MPa]</th>
<th>$\varepsilon_y$ [%]</th>
<th>$\sigma_y$ [MPa]</th>
<th>$\varepsilon_f$ [%]</th>
<th>$\sigma_f$ [MPa]</th>
<th>$E_f$ [J]</th>
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<tbody>
<tr>
<td>Healthy, male, 21 yr</td>
<td>60.0</td>
<td>4.5</td>
<td>2.6</td>
<td>10.9</td>
<td>4.0</td>
<td>28.4</td>
</tr>
<tr>
<td>Osteoarthritic, female, 65 yr</td>
<td>37.7</td>
<td>2.8</td>
<td>1.0</td>
<td>6.9</td>
<td>1.6</td>
<td>6.8</td>
</tr>
<tr>
<td>Osteoporotic, female, 85 yr</td>
<td>46.2</td>
<td>1.3</td>
<td>0.5</td>
<td>4.6</td>
<td>0.8</td>
<td>2.7</td>
</tr>
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</table>
# Stress-strain curves

Here are data from the
**LEG** of a cow,
**EAR BONES** of a fin whale
and **ANTLER** of a deer.

<table>
<thead>
<tr>
<th>Material</th>
<th>Young's modulus</th>
<th>Ultimate stress</th>
<th>Ultimate strain</th>
<th>Mineralization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cow leg</td>
<td>34</td>
<td>27</td>
<td>.2</td>
<td>86%</td>
</tr>
<tr>
<td>Fin whale ear</td>
<td>20</td>
<td>148</td>
<td>1.8</td>
<td>67%</td>
</tr>
<tr>
<td>Deer antler</td>
<td>7.2</td>
<td>158</td>
<td>11.4</td>
<td>59%</td>
</tr>
</tbody>
</table>

![Stress vs Strain Graph](image)

**Stress vs Strain Graph**

- **STRESS MPa (Force per Area)**
- **STRAIN (%change in length/ original length)**
Representative stress–strain curves for passive rat myocardium. Fiber and crossfiber stress are shown for equibiaxial strain.
The example reported on non-linear numerical analysis refers to the well known experimental tests carried out by Parfitt, (Parfitt, 1960). The tests consisted in the application of a vertical intrusive force, with a maximum magnitude of 0.03 N, on the upper incisor of human adults and in the evaluation of the corresponding vertical displacement. Parfitt’s experimental data, represented by open circles in the chart, show a strongly non-linear behaviour, with a very low initial stiffness and a hardening response, as the force increases. It is reasonable to attribute this response to a non-linear mechanical

Application of the intrusive force (a) and loading history (b) according to the experimental procedure of Parfitt.

Comparison of experimental data from Parfitt’s experiments and numerical results. Open circles represent experimental data, while a continuous line is adopted to represent numerical results.
Force-deformation response of intervertebral joints demonstrating the physiologic phase, traumatic phase, and the post-traumatic phase (post buckling range).
Molecular nanosprings in spider capture-silk threads

Force increases are exponential for stretching intact spider capture silk in air. Circles and triangles are data from two regions of capture webs.
Force Governs Function Of Proteins: The amount of energy or work done?

Nanotechnology enables us to attract individual molecules mechanically, allowing their behaviour when exposed to a mechanical force to be observed.

For a cell to survive, most eukaryotes have to be anchored to their surroundings mechanically. This is achieved by special proteins, the integrins, which cross the cell membrane. They are connected to the external periphery of the cell, the so-called extra-cellular matrix, and the cell interior, the cytoskeleton, by scaffolding proteins such as talin.

\[ \text{Talin (blue) is an important scaffolding protein that binds the contractile cytoskeleton to the extra-cellular matrix via integrins. Its structure consists of a bundle of many tightly packed alpha helixes. If a cell sticks to a surface, thus exerting forces upon the scaffolding protein, the helix bundle breaks, exposing the helixes that can only bind vinculin after this "activation".} \]
The kissing complex is an RNA tertiary structure formed by loop-loop interaction between two identical hairpins. The kissing hairpins respond to applied mechanical force like silly putty: when the force is increased slowly, the molecule is more elastic; but when the force is applied very fast, the entire structure becomes brittle.

about the thermodynamics and kinetics of the folding process
AFM Imaging of Fibrin Protofibrils

AFM spectroscopy was performed on single fibrinogen molecules (Figure D) and fibrin protofibrils (Figure E).

Force extension curve for the linear polymeric fibrin strands (protofibrils) showing a blunted `sawtooth' pattern of four triphasic coiled-coil signatures exhibiting hysteresis (extension in red and relaxation in blue). The AFM data reveal a characteristic force plateau related to the unfolding of the fibrinogen's coiled-coils.
The cycle of oxidation and reduction, causing soft/hard transitions within the molecule. The associated stretching and shrinking gives the mechanical energy. The forces are monitored by the tip of an Atomic Force Microscope, on top of the molecule. The bottom of the chain is fixed on a gold surface.
Riboswitches
Riboswitches are mRNA elements that regulate gene expression through ligand-induced conformational changes. They typically lie in the 5’ untranslated region of some genes and consist of a ligand binding aptamer domain followed by an expression platform. Aptamer conformation affects the structure adopted by the expression platform (e.g., the formation or absence of a terminator hairpin) and thus the gene’s expression. Among the simplest riboswitches are those involved in purine metabolism, which contain aptamers with a three helix junction. We have studied the aptamer of one such purine riboswitch, the pbuE adenine riboswitch.

*pbuE adenine riboswitch aptamer. ‘A’ indicates binding of the aptamer’s ligand, adenine.*
Using our single molecule dumbbell assay we are able to effectively hold the RNA of interest between two beads and exert force on it in order to probe aptamer structure. In general, we measure extension change during folding and unfolding of the aptamer and relate extension changes to numbers of nucleotides and ultimately to specific aptamer structures.

As an example, the figure below shows force-extension curves while unfolding the aptamer. In the presence of adenine (the aptamer’s ligand), the RNA is often ligand bound and stabilized; it only rips to the unfolded state at high force.
Force-extension curves showing aptamer unfolding. Without adenine, two events are seen (black), corresponding to the unfolding of hairpins P2 and P3 (inset). With adenine bound to the aptamer, large unfolding events are observed (blue), sometimes involving an intermediate state (red), that correspond to opening of the entire aptamer structure.

One can extract energies, distances, and rate constants describing aptamer folding by using force to study aptamer kinetics and equilibrium folding transitions. The adenine-induced stabilization of the closing aptamer helix, P1, was described and a quantitative energy landscape for riboswitch aptamer folding was created, dissecting the secondary and tertiary folding events of this RNA structure.
stress ($\sigma$): the distribution of force over an area
strain ($\varepsilon$): a dimensionless measure of length change
stiffness ($E$): the change in stress required for a change in strain (the slope of a stress-strain curve)

<table>
<thead>
<tr>
<th>Material</th>
<th>Young's Modulus (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Locust cuticle</td>
<td>0.2</td>
</tr>
<tr>
<td>Rubber</td>
<td>7</td>
</tr>
<tr>
<td>Human cartilage</td>
<td>24</td>
</tr>
<tr>
<td>Human tendon</td>
<td>600</td>
</tr>
<tr>
<td>Cheap plastic</td>
<td>1,400</td>
</tr>
<tr>
<td>Plywood</td>
<td>14,000</td>
</tr>
<tr>
<td>Human bone</td>
<td>21,000</td>
</tr>
<tr>
<td>Glass</td>
<td>70,000</td>
</tr>
<tr>
<td>Brass</td>
<td>120,000</td>
</tr>
<tr>
<td>Iron</td>
<td>210,000</td>
</tr>
<tr>
<td>Diamond</td>
<td>1,200,000</td>
</tr>
</tbody>
</table>
Potential energy

The work of the force of gravity $W$ during any displacement:

$$U_{1 \to 2} = W y_1 - W y_2$$

Potential energy due to force of gravity (weight):

$$V_g = W y$$

If potential energy increases (in $2$), work is negative, if potential energy increases (in $2$), work is positive.

Gravitational force:

$$V_g = - \frac{GMm}{r}$$

The work of the force exerted by a spring on the body:

$$U_{1 \to 2} = \frac{1}{2} k x_1^2 - \frac{1}{2} k x_2^2$$

Potential energy of the body with respect to the elastic force $F$:

$$V_e = \frac{1}{2} k x^2$$

$$U_{1 \to 2} = V_1 - V_2$$

These forces are independent of path: conservative
• If the weight is included as a force, it does work.
• If the weight is included in energy equation as an energy term, it only affects as a potential energy element. No work done due to elevation change.

\[ T_1 + U_{1 \rightarrow 2} = T_2 \]
Conservative Forces

- Concept of potential energy can be applied if the work of the force is independent of the path followed by its point of application.
  \[ U_{1\rightarrow 2} = V(x_1, y_1, z_1) \rightarrow V(x_2, y_2, z_2) \]
  Such forces are described as *conservative forces*.

- For any conservative force applied on a closed path, \( \int \vec{F} \cdot d\vec{r} = 0 \)

- Elementary work corresponding to displacement between two neighboring points,
  \[ dU = V(x, y, z) \rightarrow V(x + dx, y + dy, z + dz) \]
  \[ = -dV(x, y, z) \]
  \[ F_x dx + F_y dy + F_z dz = -\left( \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right) \]
  \[ \vec{F} = -\left( \frac{\partial V}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial V}{\partial z} \right) = -\text{grad}V \]
Friction forces are non conservative, the work of a friction force cannot be expressed as a change in potential energy. So friction force decreases the total mechanical energy of the system.
The power developed by a machine is defined as the time rate at which work is done:

\[
\text{Power} = \frac{dU}{dt} = \mathbf{F} \cdot \mathbf{v}
\]

where \( \mathbf{F} \) is the force exerted on the particle and \( \mathbf{v} \) is the velocity of the particle. The *mechanical efficiency*, denoted by \( \eta \), is expressed as

\[
\eta = \frac{\text{power output}}{\text{power input}}
\]
Force-Velocity Relationship

- Muscle force decreases as the velocity of contraction increases (Hill, 1938)
  - only true for concentric contraction
- Muscle force decreases with increased velocity of contraction during concentric contraction whereas it increases with increased velocity of contraction during eccentric contraction.

- Eccentric strength of a muscle can exceed isometric strength by a factor of 1.5 to 2.0, but this is true only under electric stimulation of the motor neuron.
- does NOT indicate that the muscle cannot generate strong force at a fast speed
  - maximum strength can be generated either by recruitment of more motor unit or by increase in muscle length
Factors Affecting Muscle Strength

Body Temperature
• Muscle function is most efficient at 38.5°C (101°F).
• Elevated muscle temperature → shift in force-velocity curve
  • Increased maximum isometric tension
    • Nerve conduction velocity → frequency of stimulation → muscle force
    • Enzyme activity → efficiency of muscle contraction
  • Elasticity of collagen → extensibility of muscle → muscle force
  • Increased maximum velocity of muscle shortening
    • Requiring less motor unit to sustain a given load
• Body temperature too high → heat exhaustion or heat stroke

![Shift in Force-Velocity Curve at Slightly Elevated Temperature](image)
velocity-force curves
- generated from series of isotonic contractions
- force and velocity are inversely related such that at zero (0) velocity maximum force is generated, and at maximum velocity zero (0) force is generated
- power output = force x velocity (rate of doing work)
  - measured in watts (1N x 1m/s)
  - is maximized at about 30% of maximum force
Sample Problem 13.1

SOLUTION:
- Evaluate the change in kinetic energy.
- Determine the distance required for the work to equal the kinetic energy change.

1800 kg
An automobile weighing 4000 lb is driven down a 5° incline at a speed of 96.54 km/h. 675 kg * 9.81 = 6621.75 N

Determine the distance traveled by the automobile as it comes to a stop.
SOLUTION:

- Evaluate the change in kinetic energy.

\[ v_2 = 0 \quad T_2 = 0 \]

- Determine the distance required for the work to equal the kinetic energy change.

\[ 1800 \text{ kg} \times 9.81 = 17658 \text{ N} \]

\[ 675 \text{ kg} \times 9.81 = 6621.75 \text{ N} \]
N = W \cos \theta = 17658 \times \cos 5^\circ = 17591 \text{ N} \\
F_k = \mu_k N = 6621.75 \text{ N} \\
\mu_k = 0.37 \\

clc \\
clear all \\
theta = 5 \times \pi / 180; \\
g = 9.81; \\
m = 1800; \\
v = 96.54 / 3.6; \\
fk = 6621.75; \\
N = m \times g \times \cos (\theta); \\
nuk = fk / N; \\
T1 = 1 / 2 \times m \times v^2; \\
x = T1 / (fk - m \times g \times \sin (\theta))
clc
clear all
format long g
thetaold=5*pi/180;
g=9.81;
m=1800;
v=96.54/3.6;
fkold=6621.75;
norold=m*g*cos(thetaold);
nuk=fkold/norold;
sonuc=[];
n=0;
for theta=0:5:90;
    thetanew=theta*pi/180;
    fknew=nuk*m*g*cos(thetanew);
    T1=1/2*m*v^2;
    syms x
    xx=T1+(-fknew+m*g*sin(thetanew))*x;
    xxx=eval(solve(xx));
    n=n+1;
    sonuc(n,1)=theta;
    sonuc(n,2)=xxx;
end
sonuc
\[ v_f^2 = v_i^2 + 2ax \]

\[ a = \frac{F}{m} = \frac{1800 \times 9.81 \sin 5 - 6622.5}{1800} = \frac{1539 - 6622.5}{1800} = -2.82 \frac{m}{s^2} \]

\[ x = \frac{v_f^2 - v_i^2}{2a} = -\frac{v_i^2}{2a} = \frac{6.82}{2 \times 2.82} = 125.4 m \]
Two blocks are joined by an inextensible cable as shown. If the system is released from rest, determine the velocity of block A after it has moved 2 m. Assume that the coefficient of friction between block A and the plane is $\mu_k = 0.25$ and that the pulley is weightless and frictionless.

SOLUTION:

- Apply the principle of work and energy separately to blocks A and B.
- When the two relations are combined, the work of the cable forces cancel. Solve for the velocity.
SOLUTION:
- Apply the principle of work and energy separately to blocks A and B.
When the two relations are combined, the work of the cable forces cancel. Solve for the velocity.

\[ v = 4.43 \text{ m/s} \]
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>spring deflection</td>
<td>0.225 m</td>
<td>-0.000</td>
<td>0.225</td>
</tr>
<tr>
<td>spring constant (kN/m)</td>
<td>20</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The relationship between work and potential energy is combined with the relationship between work and kinetic energy. This is the principle of conservation of energy, which states that when a particle moves under the action of conservative forces, the sum of its kinetic and potential energies remains constant (summation is called the total mechanical energy, $E$). The application of this principle facilitates the solution of problems involving only conservative forces.

Friction forces are non conservative, the work of a friction force cannot be expressed as a change in potential energy. So friction force decreases the total mechanical energy of the system. Mechanical energy is dissipated by friction into thermal energy. Total energy is constant.

**Kinetic energy**

**Elastic potential energy**

**Gravitational potential energy**
CONSERVATION OF ENERGY

WORK - ENERGY
Conservation of Energy

- Work of a conservative force,
  \[ U_{1\rightarrow2} = V_1 - V_2 \]

- Concept of work and energy,
  \[ U_{1\rightarrow2} = T_2 - T_1 \]

- Follows that
  \[ T_1 + V_1 = T_2 + V_2 \]

  \[ E = T + V = \text{constant} \]

- When a particle moves under the action of conservative forces, the total mechanical energy is constant.

\[
\begin{align*}
T_1 &= 0 \quad V_1 = W\ell \\
T_1 + V_1 &= W\ell \\
T_2 &= \frac{1}{2}mv_2^2 = \frac{1}{2}W g \ell = W\ell \quad V_2 = 0 \\
T_2 + V_2 &= W\ell 
\end{align*}
\]
\[ T_1 + V_{e1} + V_{g1} = T_2 + V_{e2} + V_{g2} \]

\[ T_1 = 0, T_2 = 0, V_{e1} = 0, V_{e2} = \frac{1}{2} k\delta^2, V_{g1} = mg \quad h + \delta, \quad V_{g2} = 0 \]

\[ T_1 = 0, T_2 = 0, V_{e1} = 0, V_{e2} = \frac{1}{2} k\delta^2, V_{g1} = mgh, V_{g2} = -mg\delta \]
\[ T_1 + V_{e1} + V_{g1} = T_2 + V_{e2} + V_{g2} \]

\[ T_1 = 0, T_2 = 0, V_{e1} = \frac{1}{2} ky_1^2, V_{e2} = \frac{1}{2} k \ y_1 + y_2 \ y_1 + y_2^2, V_{g1} = mg \ h + y_2, V_{g2} = 0 \]
A spring is used to stop a 60 kg package which is sliding on a horizontal surface. The spring has a constant $k = 20 \text{ kN/m}$ and is held by cables so that it is initially compressed 120 mm. The package has a velocity of 2.5 m/s in the position shown and the maximum deflection of the spring is 40 mm.

Determine $(a)$ the coefficient of kinetic friction between the package and surface and $(b)$ the velocity of the package as it passes again through the position shown.

**SOLUTION:**

- Apply the principle of work and energy between the initial position and the point at which the spring is fully compressed and the velocity is zero. The only unknown in the relation is the friction coefficient.
- Apply the principle of work and energy for the rebound of the package. The only unknown in the relation is the velocity at the final position.
SOLUTION:

- Apply principle of work and energy between initial position and the point at which spring is fully compressed.
• Apply the principle of work and energy for the rebound of the package.

$v_3 = 1.103 \text{ m/s}$
\[ \mu_k = 0.2 \]
\[ l = 0.48m \]
\[ T_1 + V_{e1} + U_{1\rightarrow 2} = T_2 + V_{e2} \]
\[ 187.5 + 0 - 0.2 \cdot 60 \cdot 9.81 \cdot 0.48 + x = 0 + \frac{1}{2} \cdot 20000 \cdot x^2 \]

syms x
Solve('10000*x^2-187.5+0.2*60*9.81*(0.48+x)=0')
ans =

.10871803568810306079761579826117
-.12049003568810306079761579826117
\( \mu_k = 0.2 \)

\( l = 0.60m \)

\( T_1 + V_{e1} + U_{1\rightarrow 2} = T_2 + V_{e2} \)

\[ 187.5 + 0 - 0.2 \cdot 60 \cdot 9.81 \cdot 0.60 + x = 0 + \frac{1}{2} \cdot 20000 \cdot x^2 \]

\text{syms} \ x
\text{Solve}('10000\cdot x^2-187.5+0.2\cdot 60\cdot 9.81\cdot (0.60+x)=0')
\text{ans} = 

\[ 0.10237962241080960968032615466015 \]

\[ -0.11415162241080960968032615466015 \]
A 900 kg car starts from rest at point 1 and moves without friction down the track shown.

Determine:

a) the force exerted by the track on the car at point 2, and

b) the minimum safe value of the radius of curvature at point 3.

SOLUTION:

• Apply principle of work and energy to determine velocity at point 2.
• Apply Newton’s second law to find normal force by the track at point 2.
• Apply principle of work and energy to determine velocity at point 3.
• Apply Newton’s second law to find minimum radius of curvature at point 3 such that a positive normal force is exerted by the track.
SOLUTION:

- Apply principle of work and energy to determine velocity at point 2.

\[ T_1 = 0 \quad T_2 = \frac{1}{2} m v_2^2 = \frac{1}{2} m v_2^2 \]

\[ U_{1\rightarrow2} = +W \quad 12 \text{ m} \]

\[ T_1 + U_{1\rightarrow2} = T_2 : \quad 0 + W \quad 12 \text{ m} = \frac{1}{2} m v_2^2 \]

\[ v_2^2 = 2 \quad 12 \text{ m} \quad g = 2 \quad 12 \text{ m} \quad 9.81 \text{ m/s}^2 \quad v_2 = 15.3441 \text{ m/s} \]

- Apply Newton’s second law to find normal force by the track at point 2.

\[ + \sum F_n = m a_n : \]

\[ -W + N = m a_n = m \frac{v_2^2}{\rho_2} = m \frac{2 \quad 12 \text{ m} \quad g}{6 \text{ m}} \]

\[ N = 4500 \text{ kg} \times 9.81 \text{ m/s}^2 \]

\[ N = 44145 \text{ N} \]
• Apply principle of work and energy to determine velocity at point 3.

\[ T_1 + U_{1 \rightarrow 3} = T_3 \quad 0 + W \ 12 - 4.5 \text{ m} = \frac{1}{2}mv_3^2 \]

\[ v_3^2 = 2 \ 7.5 \text{ m} \ g = 2 \ 7.5 \text{ m} \ 9.81 \text{ m/s}^2 \ v_3 = 12.1305 \text{ m/s} \]

• Apply Newton’s second law to find minimum radius of curvature at point 3 such that a positive normal force is exerted by the track.

\[ + \sum F_n = ma_n : \]

\[ W = ma_n \]

\[ = m \frac{v_3^2}{\rho_3} = m \frac{12.1305^2}{\rho_3} \]

\[ \rho_3 = \frac{m \times 12.1305^2}{mg} \]

\[ \rho_3 = 15 \text{ m} \]
The dumbwaiter $D$ and its load have a combined weight of 600 lb, while the counterweight $C$ weighs 800 lb.

Determine the power delivered by the electric motor $M$ when the dumbwaiter (a) is moving up at a constant speed of $2.4384$ m/s and (b) has an instantaneous velocity of 8 ft/s and an acceleration of $2.5$ ft/s$^2$, both directed upwards.

**SOLUTION:**

- In the first case, bodies are in uniform motion. Determine force exerted by motor cable from conditions for static equilibrium.
- In the second case, both bodies are accelerating. Apply Newton’s second law to each body to determine the required motor cable force.

Force exerted by the motor cable has same direction as the dumbwaiter velocity.

Power delivered by motor is equal to $Fv_D$, $v_D = 8$ ft/s.

$270 \times 9.81 = 2676.2$ N

$2.4384$ m/s

$0.762$ m/s$^2$
• In the first case, bodies are in uniform motion. Determine force exerted by motor cable from conditions for static equilibrium.

Free-body C:
\[ + \sum F_y = 0 : \ 2T - 360 \times 9.81 = 0 \quad T = 1765.8 \text{ N} \]

Free-body D:
\[ + \sum F_y = 0 : \ F + T - 270 \times 9.81 = 0 \]
\[ F = 270 \times 9.81 - T = 270 \times 9.81 - 1765.8 = 882.9 \text{ N} \]
\[ Power = Fv_D = 882.9 \text{ N} \times 2.4384 \text{ m/s} = 2152.86 \text{ W} = 2.153 \text{ kW} \]

\[ Power = 2152.864 \text{ W} \times \frac{1 \text{ hp}}{0.746 \text{ kW}} = 2.89 \text{ hp} \]
In the second case, both bodies are accelerating. Apply Newton’s second law to each body to determine the required motor cable force.

\[ a_D = 0.762 \text{ m/s}^2 \uparrow \quad a_C = -\frac{1}{2}a_D = 0.381 \text{ m/s}^2 \downarrow \]

Free-body C:

\[ + \downarrow \sum F_y = m_C a_C : 360 \times 9.81 - 2T = 360 \times 0.381 \quad T = 1697.22 \text{ N} \]

Free-body D:

\[ + \uparrow \sum F_y = m_D a_D : F + T - 270 \times 9.81 = 270 \times 0.762 \quad F = 1157.22 \text{ N} \]

\[ Power = Fv_D = 1157.2 \text{ N} \times 2.4384 \text{ m/s} = 2821.77 \text{ W} \]

\[ Power = 2821.77 \text{ W} \times \frac{1 \text{ hp}}{0.746 \text{kW}} = 3.78 \text{ hp} \]
When a particle moves under a central force $\mathbf{F}$, its angular momentum about the center of force $O$ remains constant. If the central force $\mathbf{F}$ is also conservative, the principles of conservation of angular momentum and conservation of energy can be used jointly to analyze the motion of the particle. For the case of oblique launching, we have

$$ (H_O)_0 = H_O : \quad r_0 m v_0 \sin \phi_0 = r m v \sin \phi $$

where $m$ is the mass of the vehicle and $M$ is the mass of the earth.
• Given \( r \), the equations may be solved for \( v \) and \( \phi \).

• At minimum and maximum \( r \), \( \phi = 90^\circ \). Given the launch conditions, the equations may be solved for \( r_{\text{min}} \), \( r_{\text{max}} \), \( v_{\text{min}} \), and \( v_{\text{max}} \).
A 9 kg collar slides without friction along a vertical rod as shown. The spring attached to the collar has an undeflected length of 10 cm and a constant of 530 N/m.

If the collar is released from rest at position 1, determine its velocity after it has moved 15 cm to position 2.

**SOLUTION:**

- Apply the principle of conservation of energy between positions 1 and 2.
- The elastic and gravitational potential energies at 1 and 2 are evaluated from the given information. The initial kinetic energy is zero.
- Solve for the kinetic energy and velocity at 2.

$$ r = 20 + i \times 15 $$

$$ r = 20.0000 + 15.0000i $$

$$ \text{angle}(r) \times \frac{180}{\pi} $$

$$ \text{ans} = 36.8699 $$

$$ \text{abs}(r) $$

$$ \text{ans} = 25 $$
SOLUTION:
• Apply the principle of conservation of energy between positions 1 and 2.
Position 1:

Conservation of Energy:

\[ v_2 = 4.91 \text{ ft/s} \]
>> sqrt((2.65+7.281)/4.5)
ans =
   1.4856

syms v
Solve('2.65=4.5*v^2-7.281')

ans =

   -1.4855601263122569718162267342793
   1.4855601263122569718162267342793
>> vpa(factorial(170),307)

ans =

72574156153079940453996357155895914678961841172422578034055442117
55693246215271577444614997868077640013184176271985826801597743247
24797907799533661942998068579328576805336088611214982543708135636
56990432878846140027884906945304696617530078018969625637211046192
42357348735986883814984039817295623520648167424

>> factorial(170)

ans =

7.2574e+306
.
.
syms x

int ('sin(2*x)^2/cos(2*x)',x)

ans =

-1/2*sin(2*x)+1/2*log(sec(2*x)+tan(2*x))

diff('sin(2*x)^2/cos(2*x)',x)

ans =

4*sin(2*x)+2*sin(2*x)^3/cos(2*x)^2
r = -400 - 400 * sin(30 * pi/180) - 400 * cos(30 * pi/180) * i

ans = 292.8203

ans = -150.0000
>> int('1/sqrt(2.754*s)')

ans =

1.2051692101036454000002972917337*s^(1/2)
Sample Problem 13.7

The 0.5 lb pellet is pushed against the spring and released from rest at A. Neglecting friction, determine the smallest deflection of the spring for which the pellet will travel around the loop and remain in contact with the loop at all times.

SOLUTION:

- Since the pellet must remain in contact with the loop, the force exerted on the pellet must be greater than or equal to zero. Setting the force exerted by the loop to zero, solve for the minimum velocity at D.

- Apply the principle of conservation of energy between points A and D. Solve for the spring deflection required to produce the required velocity and kinetic energy at D.
SOLUTION:

- Setting the force exerted by the loop to zero, solve for the minimum velocity at \( D \).
  
  \[ + \sum F_n = ma_n : \]

- Apply the principle of conservation of energy between points \( A \) and \( D \).

\[ x = 0.3727 \text{ ft} = 4.47 \text{ in.} \]
A satellite is launched in a direction parallel to the surface of the earth with a velocity of 36900 km/h from an altitude of 500 km.

Determine (a) the maximum altitude reached by the satellite, and (b) the maximum allowable error in the direction of launching if the satellite is to come no closer than 200 km to the surface of the earth.

**SOLUTION:**

- For motion under a conservative central force, the principles of conservation of energy and conservation of angular momentum may be applied simultaneously.
- Apply the principles to the points of minimum and maximum altitude to determine the maximum altitude.
- Apply the principles to the orbit insertion point and the point of minimum altitude to determine maximum allowable orbit insertion angle error.
Apply the principles of conservation of energy and conservation of angular momentum to the points of minimum and maximum altitude to determine the maximum altitude.

Conservation of energy:

\[ T_A + V_A = T_{A'} + V_{A'} \]
\[ \frac{1}{2} m v_0^2 - \frac{GMm}{r_0} = \frac{1}{2} m v_1^2 - \frac{GMm}{r_1} \]

Conservation of angular momentum:

\[ r_0 m v_0 = r_1 m v_1 \]
\[ v_1 = v_0 \frac{r_0}{r_1} \]

Combining,

\[ \frac{1}{2} v_0^2 \left(1 - \frac{r_0^2}{r_1^2} \right) = \frac{GM}{r_0} \left(1 - \frac{r_0}{r_1} \right) \]
\[ 1 + \frac{r_0}{r_1} = \frac{2GM}{r_0 v_0^2} \]
\[ r_0 = 6370 \text{km} + 500 \text{km} = 6870 \text{km} \]
\[ v_0 = 36900 \text{km/h} = 10.25 \times 10^6 \text{ m/s} \]
\[ GM = gR^2 = (9.81 \text{ m/s}^2)(3.7 \times 10^6 \text{ m}^2) = 398 \times 10^{12} \text{ m}^3/\text{s}^2 \]
\[ r_1 = 60.4 \times 10^6 \text{ m} = 60400 \text{ km} \]
• Apply the principles to the orbit insertion point and the point of minimum altitude to determine maximum allowable orbit insertion angle error.

Conservation of energy:

\[ T_0 + V_0 = T_A + V_A \]

\[ \frac{1}{2} m v_0^2 - \frac{GMm}{r_0} = \frac{1}{2} m v_{\text{max}}^2 - \frac{GMm}{r_{\text{min}}} \]

Conservation of angular momentum:

\[ r_0 m v_0 \sin \phi_0 = r_{\text{min}} m v_{\text{max}} \]

\[ v_{\text{max}} = v_0 \sin \phi_0 \frac{r_0}{r_{\text{min}}} \]

Combining and solving for \( \sin \phi_0 \):

\[ \sin \phi_0 = 0.9801 \]

\[ \phi_0 = 90^\circ \pm 11.5^\circ \]

**allowable error = \pm 11.5^\circ**
The linear momentum of a particle is defined as the product \( m \vec{v} \) of the mass \( m \) of the particle and its velocity \( \vec{v} \). From Newton’s second law, \( \vec{F} = m \vec{a} \), we derive the relation

\[
\frac{\vec{F}}{d\vec{t}} = \frac{d}{dt} (m \vec{v}) = d (m \vec{v}) = \int_{t_1}^{t_2} \vec{F} dt = m \vec{v}_2 - m \vec{v}_1
\]

where \( m\vec{v}_1 \) and \( m\vec{v}_2 \) represent the momentum of the particle at a time \( t_1 \) and a time \( t_2 \), respectively, and where the integral defines the linear impulse of the force \( \vec{F} \) during the corresponding time interval. Therefore,

\[
\text{Imp}_{1 \to 2} = \int_{t_1}^{t_2} \vec{F} dt = i \int_{t_1}^{t_2} F_x dt + j \int_{t_1}^{t_2} F_y dt + k \int_{t_1}^{t_2} F_z dt
\]

N.s=(kg.m/s²).s=kg.m/s

lb.s

The final momentum \( m\vec{v}_2 \) of the particle can be obtained by adding vectorially its initial momentum \( m\vec{v}_1 \) and the impulse of the force \( \vec{F} \) during the time interval considered.

\[
m\vec{v}_1 + \text{Imp}_{1 \to 2} = m\vec{v}_2
\]

which expresses the principle of impulse and momentum for a particle.
Impulsive Motion

- Force acting on a particle during a very short time interval that is large enough to cause a significant change in momentum is called an *impulsive force*.

- When impulsive forces act on a particle, 
  \[ m\ddot{v}_1 + \sum \vec{F} \Delta t = m\ddot{v}_2 \]

- When a baseball is struck by a bat, contact occurs over a short time interval but force is large enough to change sense of ball motion.

- *Nonimpulsive forces* are forces for which \( \vec{F} \Delta t \) is small and therefore, may be neglected.
When the particle considered is subjected to several forces, the sum of the impulses of these forces should be used;

\[ m\mathbf{v}_1 + \sum \text{Imp}_{1\rightarrow2} = m\mathbf{v}_2 \]

Since vector quantities are involved, it is necessary to consider their \( x \) and \( y \) components separately.

In the case of the impulsive motion of several particles, we write

\[ \sum m\mathbf{v}_1 + \sum \text{Imp}_{1\rightarrow2} = \sum m\mathbf{v}_2 \]

The impulses of the forces of action and reaction cancel out and only the impulses of the external forces need to be considered.

\[ \sum m\mathbf{v}_1 = \sum m\mathbf{v}_2 \]

If there is no external force, the total momentum of the particles is conserved.

\[ m_A \mathbf{v}_A = 0 \quad \text{and} \quad m_B \mathbf{v}_B = 0 \]

\[ \sum m\mathbf{v}_1 = \sum m\mathbf{v}_2 \Rightarrow 0 = m_A \mathbf{v}_A' + m_B \mathbf{v}_B' \]
The method of impulse and momentum is effective in the study of impulsive motion of a particle, when very large forces, called impulsive forces, are applied for a very short interval of time $\Delta t$, since this method involves impulses $F \Delta t$ of the forces, rather than the forces themselves. Neglecting the impulse of any nonimpulsive force, we write

$$m\mathbf{v}_1 + \sum F \Delta t = m\mathbf{v}_2$$

where the second term involves only impulsive, external forces.

Any force which is not impulsive force may be neglected.

Nonimpulsive forces include the weight of the body, the force exerted by a spring, or other small forces compared with an impulsive force.

Unknown reactions may or may not be impulsive, so they should be included.

In the case of the impulsive motion of several particles

$$\sum m\mathbf{v}_1 + \sum F \Delta t = \sum m\mathbf{v}_2$$

where the second term involves only impulsive, external forces.
\[mv_0 = 1000\text{kg} \times 25\text{m/s} = 25000 \text{kgm/s}\]

\[mv_t = 40000\text{kg} \times 25\text{m/s} = 1000000 \text{kgm/s}\]

\[\Delta t = 0.25\text{s}\]

\[mv_0 - F\Delta t = 0\]

\[F = 1000000\text{N}\]
Biomechanics in the Comics
• Fell 300 feet
  • 91.4 m
• Hit traveling 95 mph
  • 41.84 m/s
• Weight: 110 lb
  • 50 kg
• Brought to rest
  • A) 0.5 seconds
  • B) 0.1 seconds
• How large was the average force?

\[ m v_0 - F \Delta t = 0 \]

\[ F = 100000 N \]
Different magnifications of the MEMS substrate: **a)** a cut-away drawing showing the lever, the pad, and the well (bar = 10 µm); **b)** two neighboring pads (bar = 10 µm); **c)** arrays of beams (bar = 1 mm); the white square indicates the region shown in b; **d)** force diagram explaining how the cell force is calculated; **e)** micrographs (in 10 min intervals) and traction force generated at or near the leading edge. (adapted from Galbraith & Sheetz, 1997).
An automobile weighing 4000 lb is driven down a $5^\circ$ incline at a speed of 96.54 km/h or 60 mi/h when the brakes are applied, causing a constant total braking force of 1500 lb.

SOLUTION:

- Apply the principle of impulse and momentum. The impulse is equal to the product of the constant forces and the time interval.

\[
\frac{1800 \text{ kg}}{96.54 \text{ km/h}} = 6621.75 \text{ N}
\]

Determine the time required for the automobile to come to a stop.
SOLUTION:

• Apply the principle of impulse and momentum.

\[ m\vec{v}_1 + \sum \text{Imp}_{1\rightarrow 2} = m\vec{v}_2 \]

Taking components parallel to the incline,

\[ mv_1 + W \sin 5^\circ \ t - Ft = 0 \]

\[ 1800 \text{kg} \ \ 26.82 \text{m/s} \ + \ 1800 \times 9.81 \sin 5^\circ \ t - 6621.75t = 0 \]

From work and energy solution

\[ x = 125.4 \text{m} \]

\[ a = \frac{v^2}{2x} = \frac{26.82^2}{2 \times 125.4} = 2.87 \text{m/s}^2 \]

\[ t = \sqrt{\frac{2x}{a}} = 9.35 \text{s} \]

\[ t = 9.49 \text{s} \]
A 120 gr baseball is pitched with a velocity of 24 m/s. After the ball is hit by the bat, it has a velocity of 36 m/s in the direction shown. If the bat and ball are in contact for 0.015 s, determine the average impulsive force exerted on the ball during the impact.
SOLUTION:
• Apply the principle of impulse and momentum in terms of horizontal and vertical component equations.

\[ m\vec{v}_1 + \textbf{Imp}_{1 \rightarrow 2} = m\vec{v}_2 \]

\( x \) component equation:

\[ -mv_1 + F_x \Delta t = mv_2 \cos 40^\circ \]

\(-0.12 \text{kg} \cdot 24 \text{m/s} + F_x \cdot 0.015 = 0.12 \cdot 36 \cos 40^\circ \)

\[ F_x = 412.6 \text{ N} \]

\( y \) component equation:

\[ 0 + F_y \Delta t = mv_2 \sin 40^\circ \]

\[ F_y \cdot 0.015 = 0.12 \cdot 36 \sin 40^\circ \]

\[ F_y = 185 \text{ N} \]

\[ \vec{F} = 412.6 \text{ N} \hat{i} + 185 \text{ N} \hat{j}, \quad F = 452 \text{ N}, \quad \theta = 24.15^\circ \]
SOLUTION:
- Apply the principle of impulse and momentum in terms of horizontal and vertical component equations.

**x component equation:**

\[ \vec{F} = 412.6 \, \text{N} \, \hat{i} + 185 \, \text{N} \, \hat{j}, \quad F = 452 \, \text{N}, \quad \theta = 24.15^\circ \]
Sample Problem 13.12

A 10 kg package drops from a chute into a 24 kg cart with a velocity of 3 m/s. Knowing that the cart is initially at rest and can roll freely, determine (a) the final velocity of the cart, (b) the impulse exerted by the cart on the package, and (c) the fraction of the initial energy lost in the impact.

SOLUTION:

• Apply the principle of impulse and momentum to the package-cart system to determine the final velocity.
• Apply the same principle to the package alone to determine the impulse exerted on it from the change in its momentum.
SOLUTION:
• Apply the principle of impulse and momentum to the package-cart system to determine the final velocity.

\[ m_p \vec{v}_1 + \sum \text{Imp}_{1 \rightarrow 2} = (n_p + m_c) \vec{v}_2 \]

\[ x \text{ components: } m_p v_1 \cos 30^\circ + 0 = (n_p + m_c) \vec{v}_2 \]

\[ (0 \text{ kg}) (3 \text{ m/s}) \cos 30^\circ = (0 \text{ kg} + 25 \text{ kg}) \vec{v}_2 \]

\[ v_2 = 0.742 \text{ m/s} \]
• Apply the same principle to the package alone to determine the impulse exerted on it from the change in its momentum.

\[ m_p \vec{v}_1 + \sum \text{Imp}_{1 \rightarrow 2} = m_p \vec{v}_2 \]

**x components:**
\[ m_p v_1 \cos 30^\circ + F_x \Delta t = m_p v_2 \]
\[ (0 \text{ kg} \times 30 \text{ m/s} \cos 30^\circ + F_x \Delta t = (0 \text{ kg} \times \vec{v}_2) \quad F_x \Delta t = -18.56 \text{ N} \cdot \text{s} \]

**y components:**
\[ -m_p v_1 \sin 30^\circ + F_y \Delta t = 0 \]
\[ -(0 \text{ kg} \times 30 \text{ m/s} \sin 30^\circ + F_y \Delta t = 0 \quad F_y \Delta t = 15 \text{ N} \cdot \text{s} \]

\[ \sum \text{Imp}_{1 \rightarrow 2} = \vec{F} \Delta t = (18.56 \text{ N} \cdot \text{s} \hat{i} + 5 \text{ N} \cdot \text{s} \hat{j}) \quad \vec{F} \Delta t = 23.9 \text{ N} \cdot \text{s} \]
>> z=-18.56+j*15

z =

   -18.56 + 15i

>> abs(z)

ans =

     23.86364599134

>> angle(z)*180/pi

ans =

  141.055215087849
To determine the fraction of energy lost,

\[ T_1 = \frac{1}{2} m_p v_1^2 = \frac{1}{2} \cdot 10 \text{ kg} \cdot 3 \text{ m/s}^2 = 45 \text{ J} \]

\[ T_2 = \frac{1}{2} m_p + m_c \ v_2^2 = \frac{1}{2} \cdot 10 \text{ kg} + 25 \text{ kg} \cdot 0.742 \text{ m/s}^2 = 9.63 \text{ J} \]

\[ \frac{T_1 - T_2}{T_1} = \frac{45 \text{ J} - 9.63 \text{ J}}{45 \text{ J}} = 0.786 \]
IMPACT

A collision between two bodies which occurs in a very small interval of time and during which the two bodies exert relatively large forces on each other is called an *impact*. The common normal to the surfaces in contact during the impact is called the *line of impact*. If the mass centers on the two colliding bodies are located on this line, the impact is a *central impact*. Otherwise, the impact is said to be *eccentric*. If the velocities of the two particles are directed along the line of impact, the impact is said to be a *direct impact*. If either or both particles move along a line other than the line of impact, the impact is said to be an *oblique impact*. 

![Diagram of impact types](image-url)
In the case of *direct central impact*, two colliding bodies $A$ and $B$ move along the *line of impact* with velocities $\mathbf{v}_A$ and $\mathbf{v}_B$, respectively. Period of deformation, velocity $\mathbf{u}$, period of restitution, then either together or separately they will move. Two equations can be used to determine their velocities $\mathbf{v}'_A$ and $\mathbf{v}'_B$ after the impact. The first represents the conservation of the total momentum of the two bodies,

$$m_A \mathbf{v}_A + m_B \mathbf{v}_B = m_A \mathbf{v}'_A + m_B \mathbf{v}'_B$$

- Bodies moving in the same straight line, $\mathbf{v}_A > \mathbf{v}_B$.
- Upon impact the bodies undergo a *period of deformation*, at the end of which, they are in contact and moving at a common velocity.
- A *period of restitution* follows during which the bodies either regain their original shape or remain permanently deformed.
- Wish to determine the final velocities of the two bodies. The total momentum of the two body system is preserved,

$$m_A \mathbf{v}_A + m_B \mathbf{v}_B = m_A \mathbf{v}'_A + m_B \mathbf{v}'_B$$

- A second relation between the final velocities is required.
\[ m_A v_A + m_B v_B = m_A v'_A + m_B v'_B \]

Since they move in the same direction,

\[ m_A v_A + m_B v_B = m_A v'_A + m_B v'_B \]

1 equation and 2 unknown velocities.

**Impulse and momentum principle during the period of deformation**

\[ m_A v_A - \int P \, dt = m_A u \]

**Impulse and momentum principle during the period of restitution**

\[ m_A u - \int R \, dt = m_A v'_A \]

\[ \int P \, dt > \int R \, dt \]

- **Period of deformation**: \( m_A v_A - \int P \, dt = m_A u \)

- **Period of restitution**: \( m_A u - \int R \, dt = m_A v'_A \)
The constant $e$ is known as the *coefficient of restitution*; its value lies between 0 and 1 and depends on the material involved. When $e = 0$, the impact is termed *perfectly plastic*; when $e = 1$, the impact is termed *perfectly elastic*.

If $e=0$, they move together with a common velocity.

If $e=1$, they move away from each other with the same initial relative velocity, total energy and total momentum is conserved.

Provided $e$ is defined as follows:

$$
e = \frac{\int R \, dt}{\int P \, dt}$$

*coefficient of restitution*
Oblique Central Impact

In the case of *oblique central impact*, the velocities of the two colliding bodies before and after impact are resolved into $n$ components along the line of impact and $t$ components along the common tangent to the surfaces in contact. The velocities in magnitude and direction are unknown. 4 unknowns and we need 4 equations. Assume perfectly smooth and frictionless surfaces.

In the $t$ direction

$$ (v_A)_t = (v'_A)_t \quad (v_B)_t = (v'_B)_t $$

In the $n$ direction

$$ m_A (v_A)_n + m_B (v_B)_n = m_A (v'_A)_n + m_B (v'_B)_n $$

$$ (v'_B)_n - (v'_A)_n = e \left[ (v_A)_n - (v_B)_n \right] $$
• Tangential momentum of ball is conserved.
  \[ \mathbf{p}_B' = \mathbf{p}_B' \]

• Total horizontal momentum of block and ball is conserved.
  \[ m_A \mathbf{p}_A - m_B \mathbf{p}_B = m_A \mathbf{p}_A' - m_B \mathbf{p}_B' \]

• Normal component of relative velocities of block and ball are related by coefficient of restitution.
  \[ \mathbf{v}_B' \mathbf{n} - \mathbf{v}_A' \mathbf{n} = e (\mathbf{v}_A \mathbf{n} - \mathbf{v}_B \mathbf{n}). \]

• Note: Validity of last expression does not follow from previous relation for the coefficient of restitution. A similar but separate derivation is required.

Impulses from internal forces along the \( n \) axis and from external force exerted by horizontal surface and directed along the vertical to the surface.

Although this method was developed for bodies moving freely before and after impact, it could be extended to the case when one or both of the colliding bodies is constrained in its motion. Final velocity of ball unknown in direction and magnitude and unknown final block velocity magnitude. Three equations required.
Problems Involving Energy and Momentum

- Three methods for the analysis of kinetics problems:
  - Direct application of Newton’s second law
  - Method of work and energy
  - Method of impulse and momentum

- Select the method best suited for the problem or part of a problem under consideration.
A ball is thrown against a frictionless, vertical wall. Immediately before the ball strikes the wall, its velocity has a magnitude $v$ and forms angle of $30^\circ$ with the horizontal. Knowing that $e = 0.90$, determine the magnitude and direction of the velocity of the ball as it rebounds from the wall.

**SOLUTION:**

- Resolve ball velocity into components normal and tangential to wall.

- Impulse exerted by the wall is normal to the wall. Component of ball momentum tangential to wall is conserved.

- Assume that the wall has infinite mass so that wall velocity before and after impact is zero. Apply coefficient of restitution relation to find change in normal relative velocity between wall and ball, i.e., the normal ball velocity.
SOLUTION:

- Resolve ball velocity into components parallel and perpendicular to wall.
  \[ v_n = v \cos 30° = 0.866v \quad v_t = v \sin 30° = 0.500v \]

- Component of ball momentum tangential to wall is conserved.
  \[ v'_t = v_t = 0.500v \]

- Apply coefficient of restitution relation with zero wall velocity.
  \[ 0 - v'_n = e (v_n - 0) \]
  \[ v'_n = -0.9(0.866v) = -0.779v \]

\[
\frac{T_2 - T_1}{T_1} = \frac{0.926^2 - 1}{1} = 0.142
\]

\[
\vec{v}' = -0.779v \hat{\lambda}_n + 0.500v \hat{\lambda}_t
\]

\[
v' = 0.926v \quad \tan^{-1}\left(\frac{0.779}{0.500}\right) = 32.7°
\]
Sample Problem 13.15

The magnitude and direction of the velocities of two identical frictionless balls before they strike each other are as shown. Assuming $e = 0.9$, determine the magnitude and direction of the velocity of each ball after the impact.

SOLUTION:

- Resolve the ball velocities into components normal and tangential to the contact plane.
- Tangential component of momentum for each ball is conserved.
- Total normal component of the momentum of the two ball system is conserved.
- The normal relative velocities of the balls are related by the coefficient of restitution.
- Solve the last two equations simultaneously for the normal velocities of the balls after the impact.
SOLUTION:

- Resolve the ball velocities into components normal and tangential to the contact plane.

\[ \vec{v}_A^\perp = v_A \cos 30^\circ = 26.0 \text{ ft/s} \quad \vec{v}_A^\parallel = v_A \sin 30^\circ = 15.0 \text{ ft/s} \]
\[ \vec{v}_B^\perp = -v_B \cos 60^\circ = -20.0 \text{ ft/s} \quad \vec{v}_B^\parallel = v_B \sin 60^\circ = 34.6 \text{ ft/s} \]

- Tangential component of momentum for each ball is conserved.

\[ \vec{p}_A' = \vec{p}_A = 15.0 \text{ ft/s} \quad \vec{p}_B' = \vec{p}_B = 34.6 \text{ ft/s} \]

- Total normal component of the momentum of the two ball system is conserved.

\[ m_A \vec{p}_A + m_B \vec{p}_B = m_A \vec{p}_A' + m_B \vec{p}_B' \]
\[ m(6.0) + m(20.0) = m\vec{p}_A + m\vec{p}_B \]
\[ \vec{p}_A' + \vec{p}_B' = 6.0 \]
• The normal relative velocities of the balls are related by the coefficient of restitution.

\[ \mathbf{v}_A' - \mathbf{v}_B' = e (\mathbf{v}_A - \mathbf{v}_B) \]

\[ = 0.90 (6.0 - 20.0) = 41.4 \]

• Solve the last two equations simultaneously for the normal velocities of the balls after the impact.

\[ \mathbf{v}_A' = -17.7 \text{ ft/s} \quad \mathbf{v}_B' = 23.7 \text{ ft/s} \]

\[ \bar{v}_A' = -17.7 \bar{l}_t + 15.0 \bar{l}_n \]

\[ v_A' = 23.2 \text{ ft/s} \quad \tan^{-1} \left( \frac{15.0}{17.7} \right) = 40.3^\circ \]

\[ \bar{v}_B' = 23.7 \bar{l}_t + 34.6 \bar{l}_n \]

\[ v_B' = 41.9 \text{ ft/s} \quad \tan^{-1} \left( \frac{34.6}{23.7} \right) = 55.6^\circ \]
Ball $B$ is hanging from an inextensible cord. An identical ball $A$ is released from rest when it is just touching the cord and acquires a velocity $v_0$ before striking ball $B$. Assuming perfectly elastic impact $(e = 1)$ and no friction, determine the velocity of each ball immediately after impact.

SOLUTION:

- Determine orientation of impact line of action.
- The momentum component of ball $A$ tangential to the contact plane is conserved.
- The total horizontal momentum of the two ball system is conserved.
- The relative velocities along the line of action before and after the impact are related by the coefficient of restitution.
- Solve the last two expressions for the velocity of ball $A$ along the line of action and the velocity of ball $B$ which is horizontal.
SOLUTION:

- Determine orientation of impact line of action.

\[ \sin \theta = \frac{r}{2r} = 0.5 \]
\[ \theta = 30^\circ \]

- The momentum component of ball A tangential to the contact plane is conserved.

\[ m\vec{v}_A + \vec{F}\Delta t = m\vec{v}'_A \]
\[ mv_0 \sin 30^\circ + 0 = m \dot{\vec{v}}'_A \]
\[ \dot{\vec{v}}'_A = 0.5v_0 \]

- The total horizontal (x component) momentum of the two ball system is conserved.

\[ m\vec{v}_A + \vec{T}\Delta t = m\vec{v}'_A + m\vec{v}'_B \]
\[ 0 = m \dot{\vec{v}}'_A \cos 30^\circ - m \dot{\vec{v}}'_A \sin 30^\circ - mv'_B \]
\[ 0 = 0.5v_0 \cos 30^\circ - \dot{\vec{v}}'_A \sin 30^\circ - v'_B \]
\[ 0.5 \dot{\vec{v}}'_A + v'_B = 0.433v_0 \]
• The relative velocities along the line of action before and after the impact are related by the coefficient of restitution.
\[
\mathbf{\xi}'_{B,N} - \mathbf{\xi}'_{A,N} = e \left( \mathbf{\xi}'_{A,N} - \mathbf{\xi}'_{B,N} \right)
\]
\[
v'_B \sin 30° - \mathbf{\xi}'_{A,N} = v_0 \cos 30° - 0
\]
\[
0.5v'_B - \mathbf{\xi}'_{A,N} = 0.866v_0
\]

• Solve the last two expressions for the velocity of ball A along the line of action and the velocity of ball B which is horizontal.
\[
\mathbf{\xi}'_{A,N} = -0.520v_0 \quad v'_B = 0.693v_0
\]
\[
\bar{v}'_A = 0.5v_0\hat{\xi}_t - 0.520v_0\hat{\xi}_n
\]
\[
v'_A = 0.721v_0 \quad \beta = \tan^{-1}\left(\frac{0.52}{0.5}\right) = 46.1°
\]
\[
\alpha = 46.1° - 30° = 16.1°
\]
\[
v'_B = 0.693v_0 \leftarrow
\]
A 30 kg block is dropped from a height of 2 m onto the 10 kg pan of a spring scale. Assuming the impact to be perfectly plastic, determine the maximum deflection of the pan. The constant of the spring is \( k = 20 \text{ kN/m} \).

**SOLUTION:**

- Apply the principle of conservation of energy to determine the velocity of the block at the instant of impact.
- Since the impact is perfectly plastic, the block and pan move together at the same velocity after impact. Determine that velocity from the requirement that the total momentum of the block and pan is conserved.
- Apply the principle of conservation of energy to determine the maximum deflection of the spring.
SOLUTION:

- Apply principle of conservation of energy to determine velocity of the block at instant of impact.

\[ T_1 = 0 \quad V_1 = W_{Ay} = \frac{1}{2}m_A \cdot \frac{3}{2} = \frac{1}{2} \cdot 0 \cdot \frac{3}{2} \quad V_2 = 0 \]

\[ T_1 + V_1 = T_2 + V_2 \]

\[ 0 + 588 J = \frac{1}{2} \cdot 0 \cdot \frac{3}{2} + 0 \quad \frac{3}{2} = 6.26 \text{ m/s} \]

- Determine velocity after impact from requirement that total momentum of the block and pan is conserved.

\[
\frac{1}{2} m_A v_A^2 + m_B v_B^2 + m_A g \Delta t + m_B g \Delta t + F_{sp} \Delta t = m_A + m_B \quad v_3
\]

\[
\frac{1}{2} \times 30 \times 6.26^2 = 587.814 \text{ J} \quad m_A \quad v_A^2 + m_B \quad v_B^2 = m_A + m_B \quad v_3
\]

\[
\frac{1}{2} \times 40 \times 4.70^2 = 441.800 \text{ J} \quad \frac{1}{2} \times 30 \times 6.26 + 0 = 30 + 10 \quad v_3 \quad v_3 = 4.70 \text{ m/s}
\]

\[
\frac{1}{2} \times 10 \times 4.70^2 = 110.450 \text{ J}
\]
Initial spring deflection due to pan weight:

\[ x_3 = \frac{W_B}{k} = \frac{0.81}{20 \times 10^3} = 4.91 \times 10^{-3} \text{ m} \]

- Apply the principle of conservation of energy to determine the maximum deflection of the spring.

\[
T_3 = \frac{1}{2} m_A + m_B \frac{v_3^2}{v_3} = \frac{1}{2} (0 + 10 \times 4.7^2) = 442 \text{ J}
\]

\[
V_3 = V_g + V_e = 0 + \frac{1}{2} k x_3^2 = \frac{1}{2} (0 \times 10^3 \times 91 \times 10^{-3})^2 = 0.241 \text{ J}
\]

\[ T_4 = 0 \]

\[
V_4 = V_g + V_e = (V_A + W_B) h + \frac{1}{2} k x_4^2
\]

\[
= -392 (4.91 \times 10^{-3}) - x_3 + \frac{1}{2} (0 \times 10^3) x_4^2
\]

\[
= -392 (4.91 \times 10^{-3}) - \frac{1}{2} (0 \times 10^3) x_4^2
\]

\[ T_3 + V_3 = T_4 + V_4 \]

\[ 442 + 0.241 = 0 - 392 (4.91 \times 10^{-3}) - \frac{1}{2} (0 \times 10^3) x_4^2 \]

\[ x_4 = 0.230 \text{ m} \]

\[ h = x_4 - x_3 = 0.230 \text{ m} - 4.91 \times 10^{-3} \text{ m} \quad \text{(or) } h = 0.225 \text{ m} \]
\[ \sum F = ma = 0 \]
\[ m_a + m_b \ g - ky = 0 \]
\[ y = \frac{m_a + m_b \ g}{k} \]
\[ v_{\text{max}} \]
\[ a=0 \]
A 25-kg block initially at rest is acted upon by a force $P$ which varies in magnitude with time as shown. The coefficients of static and kinetic friction between the block and the horizontal surface are $\mu_s = 0.5$ and $\mu_k = 0.4$. (a) Determine and plot the speed of the block as a function of $t$ for $0 \leq t \leq 16s$. (b) Determine the maximum velocity of the block and the corresponding value of $t$.

The static friction force is

$$F_s = \mu_s W = 0.5(25)(9.81) = 122.625 \text{ N}$$
50t = 122.625

\[ t = 2.45s \]

P = 122.625N

\[ F_k = 98.1N \]

\[
\int_{2.45}^{t} P - F_k \ dt = mv
\]

\[
\int_{2.45}^{t} 50t - 98.1 \ dt = 25v
\]

\[ v = 1.000t^2 - 3.9240t + 3.6113 \]

\[
25 \times 36.22 + \int_{8}^{t} 800 - 50t - 98.1 \ dt = 25v
\]

\[
a = \text{int('}50\times t-98.1',2.45,t)/25
\]

\[
> \ \text{subs}(a,8)
\]

\[ \text{ans} = 36.2193 \]

\[
b = 36.22 + \text{int('}800-50\times t-98.1',8,t)/25
\]

\[
> \ \text{eval}(b)
\]

\[ \text{ans} = -31097/250 + 7019/250 \times t - t^2 \]

\[
> \ \text{diff}(b,t)
\]

\[ \text{ans} = 7019/250 - 2 \times t \]

\[
> \ \text{subs}(b,14.03)
\]

\[ \text{ans} = 72.6774 \]

\[
> \ \text{subs}(b,16)
\]

\[ \text{ans} = 68.8280 \]
50t = 122.625
\( t = 2.45s \)
\( P = 122.625N \)
\( F_k = 98.1N \)
\[
\frac{P - F_k}{m} = a
\]
\[
\frac{50t - 98.1}{25} = a = 2t - 3.9240
\]
\[
v = 1.000*t^2 - 3.9240*t + 3.6113
\]
\[
25 \cdot 36.22 + \int_{8}^{t} 800 - 50t - 98.1 \ dt = 25v
\]
\[ F + m_1g + m_2g - \frac{m_3}{2}g = ((50 + 25) + \frac{75}{2})^2 \]

\[ 2(F + m_1g + m_2g) - m_3g = ((50 + 25) \times 2 + 75)^2 \]
\[ T_5 = 75 \times 9.81 \text{ N} = 75a = 75 \times 2 \]

\[ T_5 - 75 \times 9.81 = 75a = 75 \times 2 \]
T2 = T3 = T4 = T5/2
\[ m_2 = 50 \text{ kg} \]

\[ T_1 + m_2 g - T_2 = m_2 a \]
\[ F + m_1 g - T_1 = m_1 a \]

\[ m_1 = 25 \text{ kg} \]
clear all
clc
format long g
m1=25;m2=50;m3=75;a1=4;a2=4;a3=2;g=9.81;
km2=[1 -1 0 0;
    0 1 -1 0;
    0 0 2 -1;
    0 0 0 1];
st2=[m1*(a1-g);m2*(a2-g);0;m3*(g+a3)];
bilinmeyenler2=inv(km2)*st2

\[
\begin{align*}
F - T_1 &= m_1 a - m_1 g \\
T_1 - T_2 &= m_2 a - m_2 g \\
2T_2 - T_5 &= 0 \\
T_5 &= m_3 \cdot a_3 + m_3 g
\end{align*}
\]
A = solve('\text{F-T1}=25(4-9.81)', 'T1-T2=50(4-9.81)', '2T2-T5=0', 'T5=75(9.81+2)');

bilinmeyenler1 = eval([\text{A.F A.T1 A.T2 A.T5}'])
\[ m_w g - F = m_w a \]

**OR**

\[ m_1 g + m_w g - T_1 = m_1 a + m_w a \]

\[ m_1 = 25 \text{ kg} \]

\[ F + m_1 g - T_1 = m_1 a \]
B=solve('mw*(9.81-4)-T1=25*(4-9.81)', 'T1-T2=50*(4-9.81)', '2*T2-T5=0', 'T5=75*(9.81+2)');
bilinmeyen3=eval([B.mw*9.81 B.T1 B.T2 B.T5'])

km4=[(9.81-4) -1 0 0;
   0 1 -1 0;
   0 0 2 -1;
   0 0 0 1];
st4=[m1*(a1-g);m2*(a2-g);0;m3*(g+a3)];
bilinmeyeuler4=inv(km4)*st4;
WORK AND ENERGY METHOD

\[ T_1 + U_{1 \rightarrow 2} = T_2 \]

\[ a_3 = 2 \text{m/s}^2 \]

\[ a_1 = a_2 = 4 \text{m/s}^2 \]

\[ v_{A1} = v_{B1} = 2v_0 \]

\[ v_{C1} = v_0 \]

\[ v_{A2} = v_{B2} = 4t + 2v_0 \]

\[ v_{C2} = 2t + v_0 \]

\[ T_1 = \frac{1}{2} m_A v_{A1}^2 + \frac{1}{2} m_B v_{B1}^2 + \frac{1}{2} m_C v_{C1}^2 \]

\[ T_2 = \frac{1}{2} m_A v_{A2}^2 + \frac{1}{2} m_B v_{B2}^2 + \frac{1}{2} m_C v_{C2}^2 \]
\[ T_1 + U_{1\rightarrow 2} = T_2 \]
\[ v_{A1} = v_{B1} = 0 = v_{C1} \]
\[ v_{A2} = v_{B2} = 4t \]
\[ v_{C2} = 2t \]
\[ T_1 = 0 \]
\[ T_2 = \frac{1}{2} m_A \left( 4t^2 \right) + \frac{1}{2} m_B \left( 4t^2 \right) + \frac{1}{2} m_C \left( 2t^2 \right) \]
\[ U_{1\rightarrow 2} = F + 25 \cdot 9.81 + 50 \cdot 9.81 \left( \frac{1}{2} 4t^2 \right) - 75 \cdot 9.81 \left( \frac{1}{2} 2t^2 \right) \]
\[ F + 25 \cdot 9.81 + 50 \cdot 9.81 \left( \frac{1}{2} 4t^2 \right) - 75 \cdot 9.81 \left( \frac{1}{2} 2t^2 \right) = \frac{1}{2} m_A \left( 4t^2 \right) + \frac{1}{2} m_B \left( 4t^2 \right) + \frac{1}{2} m_C \left( 2t^2 \right) \]
\[ F + 25 \cdot 9.81 + 50 \cdot 9.81 \left( \frac{1}{2} 4t^2 \right) - 75 \cdot 9.81 \left( \frac{1}{2} 2t^2 \right) = \frac{1}{2} m_A \left( 4t^2 \right) + \frac{1}{2} m_B \left( 4t^2 \right) + \frac{1}{2} m_C \left( 2t^2 \right) \]
\[ F + 25 \cdot 9.81 + 50 \cdot 9.81 \left( \frac{1}{2} 4t^2 \right) - 75 \cdot 9.81 \left( \frac{1}{2} 2t^2 \right) = \frac{1}{2} \left( 25 \cdot 16 + 50 \cdot 16 + 75 \cdot 4 \right) \]
\[ F = \frac{1}{2} (25 + 50 + 16 + 75) \times 9.81 = \frac{1}{2} (166) \times 9.81 = 7.125 \text{N} \]
WORK AND ENERGY METHOD, Alternative way

\[ T_1 + V_{g1} + V_{e1} + U_{1\rightarrow2} = T_2 + V_{g2} + V_{e2} \]

\[ a_3 = 2\text{m/s}^2 \]

\[ a_1 = a_2 = 4\text{m/s}^2 \]

\[ v_{A1} = v_{B1} = 2v_0 \]

\[ v_{C1} = v_0 \]

\[ v_{A2} = v_{B2} = 4t + 2v_0 \]

\[ v_{C2} = 2t + v_0 \]

\[ T_1 = \frac{1}{2} m_A v_{A1}^2 + \frac{1}{2} m_B v_{B1}^2 + \frac{1}{2} m_C v_{C1}^2 \]

\[ T_2 = \frac{1}{2} m_A v_{A2}^2 + \frac{1}{2} m_B v_{B2}^2 + \frac{1}{2} m_C v_{C2}^2 \]
\[ v_{A1} = v_{B1} = 0 = v_{C1} \]
\[ v_{A2} = v_{B2} = 4t \]
\[ v_{C2} = 2t \]
\[ T_1 = 0 \]
\[ T_2 = \frac{1}{2} m_A \cdot 4t^2 + \frac{1}{2} m_B \cdot 4t^2 + \frac{1}{2} m_C \cdot 2t^2 \]
\[ V_{g1} = m_A g \cdot y_1 + m_B g \cdot y_1 + m_C g \cdot y_2 \]
\[ V_{g2} = m_A g \cdot y_1 + 2h + m_B g \cdot y_1 + 2h + m_C g \cdot y_2 - h \]
\[ U_{1 \rightarrow 2} = F \cdot 2h \]
\[ 2h = \frac{1}{2} 4t^2 \]
**IMPULSE AND MOMENTUM METHOD**

\[
\begin{align*}
\vec{m}v_1 + \vec{F}\Delta t &= \vec{m}v_2 \\
-m_Av_{A1} - m_Bv_{B1} - \frac{m_Cv_{C1}}{2} - F\Delta t - m_Ag\Delta t - m_Bg\Delta t + \frac{m_Cg}{2}\Delta t &= -m_Av_{A2} - m_Bv_{B2} - \frac{m_Cv_{C2}}{2} \\
-F\Delta t - m_Ag\Delta t - m_Bg\Delta t + \frac{m_Cg}{2}\Delta t &= -m_A2v_{C2} - m_B2v_{C2} - \frac{m_Cv_{C2}}{2} \\
-F\Delta t - m_Ag\Delta t - m_Bg\Delta t + \frac{m_Cg}{2}\Delta t &= -m_A2a_{C2}\Delta t - m_B2a_{C2}\Delta t - \frac{m_Ca_{C2}\Delta t}{2} \\
F &= m_A2a_C + m_B2a_C + \frac{m_Ca_C}{2} - m_Ag - m_Bg + \frac{m_Cg}{2} \\
F &= +25 \times 4 + 50 \times 4 + 75 / 2 \times 2 - 25 \times 9.81 - 50 \times 9.81 + \frac{75}{2} \times 9.81 \\
F &= 7.125N
\end{align*}
\]
\[ m\ddot{v}_1 + \ddot{F}\Delta t = m\ddot{v}_2 \]
\[ -m_A v_{A1} + T_1 \Delta t - F\Delta t - m_A g \Delta t = -m_A v_{A2} \]
\[ -m_B v_{B1} - T_1 \Delta t + T_2 \Delta t - m_B g \Delta t = -m_B v_{B2} \]
\[ m_C v_{C1} + T_5 \Delta t - m_C g \Delta t = m_C v_{C2} \]
\[ T_5 = 2T_2 \]
\[ T_1 \Delta t - F\Delta t - m_A g \Delta t = -m_A 4 \Delta t \]
\[ -T_1 \Delta t + T_2 \Delta t - m_B g \Delta t = -m_B 4 \Delta t \]
\[ 2T_2 \Delta t - m_C g \Delta t = m_C 2 \Delta t \]

\[
\begin{bmatrix}
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
F
\end{bmatrix}
= 
\begin{bmatrix}
-m_A & 4 - g \\
-m_B & 4 - g \\
m_C & 2 + g / 2
\end{bmatrix}
\]

\[
\text{inv}([1 0 -1; -1 1 0; 0 1 0]) \cdot [-25*(4-9.81) -50*(4-9.81) 75*(2+9.81)/2]
\]

\[
\text{ans} =
\begin{bmatrix}
152.3750 \\
442.8750 \\
7.1250
\end{bmatrix}
\]

\[
> 75/2*(9.81+2)-50*(9.81-4)-25*(9.81-4)
\]

\[
\text{ans} =
7.1250
\]
\[ m \ddot{v}_1 + \ddot{F} \Delta t = m \ddot{v}_2 \]

\[-m_U v_{U1} - m_A v_{A1} - m_B v_{B1} - \frac{m_C v_{C1}}{2} - m_U g \Delta t - m_A g \Delta t - m_B g \Delta t + \frac{m_C g}{2} \Delta t \]

\[ = -m_U v_{U2} - m_A v_{A2} - m_B v_{B2} - \frac{m_C v_{C2}}{2} \]

\[-m_U g \Delta t - m_A g \Delta t - m_B g \Delta t + \frac{m_C g}{2} \Delta t \]

\[ = -m_U 2v_{C2} - m_A 2v_{C2} - m_B 2v_{C2} - \frac{m_C v_{C2}}{2} \]

\[-m_U g \Delta t - m_A g \Delta t - m_B g \Delta t + \frac{m_C g}{2} \Delta t \]

\[ = -m_U 2a_{C2} \Delta t - m_A 2a_{C2} \Delta t - m_B 2a_{C2} \Delta t - \frac{m_C a_{C2} \Delta t}{2} \]

\[ m_U = \]
100 kg
100 kg
m = 50 kg
F = 500 N

150 kg
F = 500 N
\[
F + W_1 - W_2 = Ma = 200a \\
F = Ma = 200a
\]

\[
F + W_1 - W_2 = Ma = 300a \\
F = Ma = 300a
\]
F=100 N

M=10 kg
a=0
V=constant or zero

F2=100 N

M=10 kg
a=-2 m/s^2

ΣF=ma=F1+100=10(-2)=\ldots F1=-120 N
F1=-120 N
Newton’s 2nd law reduces to Newton’s 3rd law when the mass is negligible.
Biomechanics in the Comics

• Fell 300 feet
  • 91.4 m
• Hit traveling 95 mph
  • 41.84 m/s
• Weight: 110 lb
  • 50 kg
• Brought to rest
  • A) 0.5 seconds
  • B) 0.1 seconds
• How large was the average force?

• $F_1 = 4184 \text{ N}$
• $F_2 = 20920 \text{ N}$

• Find the distance required to stop her.
13.19

GIVEN:
Blocks released from rest; no friction

FIND:
(a) Velocity of block B after it has moved 2 m.
(b) Tension in the cable.
KINEMATICS \[ x_B = 2x_A \]
\[ v_B = 2v_A \]

A AND B
Assume B moves down
\[ u_1 = 0 \quad T_1 = 0 \]

\[ T_2 = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2 = \frac{1}{2} (2kg) \left( \frac{v_B^2 + v_B^2}{4} \right) \]

\[ T_2 = \frac{5}{4} v_B^2 \]

\[ u_{1-2} = -m_A g (\cos 30^\circ)(x_A) + m_B g (\cos 30^\circ) \kappa_B \]
\[ \kappa_B = 2m \]
\[ \kappa_A = 1m \]

\[ u_{1-2} = (2)(9.81)(\frac{\sqrt{3}}{2})[-1 + 2] \]

\[ u_{1-2} = 16.99 \text{ J} \]
Since work is positive block B does move down

\[ T_1 + U_{1-2} = T_2 \]

\[ 0 + 16.99 = \frac{5}{4} U_B^2 \]

\[ U_B^2 = 13.59 \]

\[ U_B = 3.69 \text{ m/s} \]

Down to the right
B ALONE

\[ V_1 = 0 \quad T_1 = 0 \]
\[ V_2 = 3.69 \text{ m/s, (FROM (a))} \]

\[ T_2 = \frac{1}{2} m_B V_2^2 = \frac{1}{2} (2)(3.69)^2 = 13.59 \text{ J} \]

\[ U_{1-2} = (m_B g)(\cos 30^\circ)(x_B) \cdot (T)(x_B) \]
\[ U_{1-2} = \left[ (2 \text{ kg})(9.81 \text{ m/s}^2)(\frac{\sqrt{3}}{2}) - (T) \right](2 \text{ m}) \]
\[ U_{1-2} = 33.98 - 2T \]

\[ T_1 + U_{1-2} = T_2 \quad 0 + 33.98 - 2T = 13.59 \]
\[ 2T = 33.98 - 13.59 = 20.39 \]
\[ T = 10.19 \text{ N} \]
OR

\[ m_B g \cos 30 - T = m_B a_B \]
\[ v_B^2 = 2a_B x_B \]
\[ T = m_B g \cos 30 - m_B a_B \]
\[ T = m_B g \cos 30 - m_B \frac{v_B^2}{2x_B} \]
Velocity before landing is 10 m/s
Impact time before coming to a stop = 0.22 s
mass = 85 kg

FIND

Horizontal Component of the average impulsive force on the athletes feet

\[
85 \cdot 10 \cos 35 - P_H \cdot 0.22 = 0
\]

\[
P_H = 3165 \text{N}
\]
\[ 85 \times 10 \sin 35 + W - P_v \times 0.22 = 0 \]

\[ P_v = 6006 \text{N} \]

\[ P = 6789 \text{N} \]
GIVEN:

\( m_A = 3 \text{ kg} \)

Initial velocities of disks A and B are equal and opposite of magnitude \( u_0 \).

After impact \( u_A' = 0 \)

\( e = 0.5 \). No friction.

FIND:

(a) \( m_B \)

(b) Range of values for \( m_B \) if \( e \) is unknown.
(a) **TOTAL MOMENTUM CONSERVED**

\[ \begin{align*}
    \mathbf{v}_A &= \mathbf{v}_0 \\
    \mathbf{v}_B &= -\mathbf{v}_0 \\
    \mathbf{v}'_A &= 0 \\
    \mathbf{v}'_B &= \mathbf{v}'
\end{align*} \]

\[ \begin{array}{c}
    \circ \quad \circ = \quad \circ \quad \circ \\
    A \quad B \quad A \quad B
\end{array} \]

\[ \begin{align*}
    m_A \mathbf{v}_A + m_B \mathbf{v}_B &= m_A \mathbf{v}_A' + m_B \mathbf{v}_B' \\
    (3 \text{kg})(\mathbf{v}_0) + m_B (\mathbf{v}_0) &= 0 + m_B \mathbf{v}' \\
    \mathbf{v}' &= \frac{3 \mathbf{v}_0}{m_B} - \mathbf{v}_0 \\
    \mathbf{v}' &= \mathbf{v}_0 \left( \frac{3}{m_B} - 1 \right) \quad (1)
\end{align*} \]

**RELATIVE VELOCITIES**

\[ \begin{align*}
    (\mathbf{v}'_A - \mathbf{v}'_B) &= (\mathbf{v}'_B - \mathbf{v}'_A) \\
    2 \mathbf{v}_0 \mathbf{e} &= \mathbf{v}' - \mathbf{0} \\
    \mathbf{v}' &= 2 \mathbf{v}_0 \mathbf{e} \quad (2)
\end{align*} \]

**SUBSTITUTE FOR \( \mathbf{v}' \) IN EQUATION (1), FROM (2)**

\[ \begin{align*}
    2 \mathbf{v}_0 \mathbf{e} &= \mathbf{v}_0 \left( \frac{3}{m_B} - 1 \right) \\
    \mathbf{e} &= 0.5 \quad (2)(.5) = \frac{3}{m_B} - 1
\end{align*} \]

\[ m_B = \frac{3}{2} \text{ kg} \]

(b) FROM EQ. (3)

\[ \begin{align*}
    2 \mathbf{e} + 1 &= \frac{3}{m_B} \\
    m_B &= \frac{3}{2 \mathbf{e} + 1} \\
    \mathbf{e} &= 0 \quad m_B = 3 \text{ kg} \\
    \mathbf{e} &= 1 \quad m_B = 1 \text{ kg} \\
    &\text{1 kg} < m_B < 3 \text{ kg}
\end{align*} \]
```matlab
>> ezplot('3/(2*e+1)',[0 1])
```
Problem 3, 2005-2006 Semester 2 Problem 3

\[ mg \sin \theta - \mu mg \cos \theta = ma \]
\[ g \sin \theta - \mu g \cos \theta = a \]
\[ g / 2 - 0.4 \times 0.866g = a \]

\[ F_t + m \times 0 = m \times 0 \]
\[ 2s_A + s_C = L_1 \]
\[ H - s_C + H - s_C - s_B = L_2 \]

\[ 2v_A + v_C = 0 \]
\[ -2v_C - v_B = 0 \]

\[ 4v_A + 2v_C = 0 \]
\[ 2v_C + v_B = 0 \]

\[ 4v_A - v_B = 0 \]
10 kg

5 kg