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In the current chapter, you will study the motion of systems of particles.

The effective force of a particle is defined as the product of its mass and acceleration. It will be shown that the system of external forces acting on a system of particles is equipollent with the system of effective forces of the system.

The mass center of a system of particles will be defined and its motion described.

Application of the work-energy principle and the impulse-momentum principle to a system of particles will be described. Result obtained are also applicable to a system of rigidly connected particles, i.e., a rigid body.

Analysis methods will be presented for variable systems of particles, i.e., systems in which the particles included in the system change.
Newton's third law does not mean that forces always cancel out so that nothing can ever move. If these two figure skaters, initially at rest, push against each other, they will both move.
Systems of particles, rigid parts

Systems of particles, rigid and flexible parts
More complex
transportation-related injuries and fatalities
Slip and Fall Dynamics

R Heel Strike
- Center of Gravity (CG)
- Loss of right heel traction
- R foot slides forward

Loss of Balance
- CG falls and slip dynamics induce rearward rotation
- L foot/leg cannot provide sufficient stability
- R foot continues to slide forward

Fall Back and/or to Side
- Resulting in posterior and/or lateral contact and resulting trauma (e.g., buttocks, hips, spine, head)
Molecule of the Month images have been animated to show the large and small subunits of the ribosome, and to show oxygen binding in hemoglobin.
Gatekeeper Protein (Oct 2008)
Hair Cell Function: The hair cell possesses a tapered bundle of specialized microvilli called stereocilia. The tips of the microvilli are mechanically linked by tip filaments, which are thought to be directly connected to mechanically gated cation channels in the plasma membrane covering the stereocilia. When the bundle of stereocilia is deflected in the direction towards the shorter stereocilia, tension is reduced on the tip filaments (also referred to as gating springs) and the ion channels close and the cell becomes hyperpolarized. When the bundle of stereocilia is deflected in the direction towards the longer stereocilia, the tension on the tip filaments is increased and the mechanically gated ions channels are opened, allowing potassium and calcium to enter the cell, which becomes depolarized. The state of polarization of the hair cell affects the rate of firing of the sensory neuron which innervates each hair cell.
Application of Newton’s Laws. Effective Forces

- Newton’s second law for each particle $P_i$ in a system of $n$ particles,

\[
\vec{F}_i + \sum_{j=1}^{n} \vec{f}_{ij} = m_i \vec{a}_i
\]

\[
\vec{r}_i \times \vec{F}_i + \sum_{j=1}^{n} (\vec{r}_i \times \vec{f}_{ij}) = \vec{r}_i \times m_i \vec{a}_i
\]

$\vec{F}_i$ = external force $\quad \vec{f}_{ij}$ = internal forces $\quad m_i \vec{a}_i$ = effective force

- The system of external and internal forces on a particle is *equivalent* to the effective force of the particle.

- The system of external and internal forces acting on the entire system of particles is *equivalent* to the system of effective forces.
Application of Newton’s Laws. Effective Forces

- Summing over all the elements,
  \[
  \sum_{i=1}^{n} \vec{F}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \vec{f}_{ij} = \sum_{i=1}^{n} m_i \vec{a}_i
  \]

- Since the internal forces occur in equal and opposite collinear pairs, the resultant force and couple due to the internal forces are zero,
  \[
  \sum_{i=1}^{n} (\vec{r}_i \times \vec{F}_i) + \sum_{i=1}^{n} \sum_{j=1}^{n} (\vec{r}_i \times \vec{f}_{ij}) = \sum_{i=1}^{n} (\vec{r}_i \times m_i \vec{a}_i)
  \]

- The system of external forces and the system of effective forces are \textit{equipollent} by not \textit{equivalent}.
Linear & Angular Momentum

• Linear momentum of the system of particles,

\[ \bar{L} = \sum_{i=1}^{n} m_i \ddot{v}_i \]

\[ \dot{L} = \sum_{i=1}^{n} m_i \ddot{v}_i = \sum_{i=1}^{n} m_i \dddot{a}_i \]

• Resultant of the external forces is equal to rate of change of linear momentum of the system of particles,

\[ \sum \bar{F} = \dot{L} \]

• Angular momentum about fixed point \( O \) of system of particles,

\[ \vec{H}_O = \sum_{i=1}^{n} (\vec{r}_i \times m_i \ddot{v}_i) \]

\[ \dot{H}_O = \sum_{i=1}^{n} (\vec{r}_i \times m_i \dddot{v}_i) + \sum_{i=1}^{n} (\vec{r}_i \times m_i \dddot{a}_i) \]

\[ = \sum_{i=1}^{n} (\vec{r}_i \times m_i \dddot{a}_i) \]

• Moment resultant about fixed point \( O \) of the external forces is equal to the rate of change of angular momentum of the system of particles,

\[ \sum \bar{M}_O = \dot{H}_O \]
Motion of the Mass Center of a System of Particles

• Mass center $G$ of system of particles is defined by position vector $\vec{r}_G$ which satisfies

$$m\vec{r}_G = \sum_{i=1}^{n} m_i \vec{r}_i$$

• Differentiating twice,

$$m\ddot{\vec{r}}_G = \sum_{i=1}^{n} m_i \ddot{\vec{r}}_i$$

$$m\ddot{\vec{v}}_G = \sum_{i=1}^{n} m_i \ddot{\vec{v}}_i = \vec{L}$$

$$m\ddot{\vec{a}}_G = \ddot{\vec{L}} = \sum \vec{F}$$

• The mass center moves as if the entire mass and all of the external forces were concentrated at that point.
Angular Momentum About the Mass Center

- The angular momentum of the system of particles about the mass center,

\[ \vec{H}_G' = \sum_{i=1}^{n} (\vec{r}_i' \times m_i \vec{v}_i') \]

\[ \dot{\vec{H}}_G' = \sum_{i=1}^{n} (\vec{r}_i' \times m_i \vec{a}_i') = \sum_{i=1}^{n} (\vec{r}_i' \times m_i (\vec{a}_i - \vec{a}_G)) \]

\[ = \sum_{i=1}^{n} (\vec{r}_i' \times m_i \vec{a}_i) - \left( \sum_{i=1}^{n} m_i \vec{r}_i' \right) \times \vec{a}_G \]

\[ = \sum_{i=1}^{n} (\vec{r}_i' \times m_i \vec{a}_i) = \sum_{i=1}^{n} (\vec{r}_i' \times \vec{F}_i) \]

\[ = \sum \vec{M}_G \]

- Consider the centroidal frame of reference \( Gx'y'z' \), which translates with respect to the Newtonian frame \( Oxyz \).

- The centroidal frame is not, in general, a Newtonian frame.

- The moment resultant about \( G \) of the external forces is equal to the rate of change of angular momentum about \( G \) of the system of particles.
Angular Momentum About the Mass Center

Angular momentum about $G$ of particles in their absolute motion relative to the Newtonian $Oxyz$ frame of reference.

\[
\vec{H}_G = \sum_{i=1}^{n} (\vec{r}'_i \times m_i \vec{v}_i)
\]

\[
= \sum_{i=1}^{n} (\vec{r}'_i \times m_i (\vec{v}_G + \vec{v}'_i))
\]

\[
= \left(\sum_{i=1}^{n} m_i \vec{r}'_i\right) \times \vec{v}_G + \sum_{i=1}^{n} (\vec{r}'_i \times m_i \vec{v}_i)
\]

\[
\vec{H}_G = \vec{H}'_G = \sum \vec{M}_G
\]

Angular momentum about $G$ of the particles in their motion relative to the centroidal $Gx' y' z'$ frame of reference,

\[
\vec{H}'_G = \sum_{i=1}^{n} (\vec{r}'_i \times m_i \vec{v}'_i)
\]

Angular momentum about $G$ of the particle momenta can be calculated with respect to either the Newtonian or centroidal frames of reference.
Conservation of Momentum

• If no external forces act on the particles of a system, then the linear momentum and angular momentum about the fixed point $O$ are conserved.

\[
\dot{\mathbf{L}} = \sum \mathbf{F} = 0 \quad \dot{\mathbf{H}}_O = \sum \mathbf{M}_O = 0 \\
\bar{L} = \text{constant} \quad \bar{H}_O = \text{constant}
\]

• Concept of conservation of momentum also applies to the analysis of the mass center motion,

\[
\dot{\mathbf{L}} = \sum \mathbf{F} = 0 \quad \dot{\mathbf{H}}_G = \sum \mathbf{M}_G = 0 \\
\bar{L} = m\tilde{v}_G = \text{constant} \quad \bar{H}_G = \text{constant}
\]

• In some applications, such as problems involving central forces,

\[
\dot{\mathbf{L}} = \sum \mathbf{F} \neq 0 \quad \dot{\mathbf{H}}_O = \sum \mathbf{M}_O = 0 \\
\bar{L} \neq \text{constant} \quad \bar{H}_O = \text{constant}
\]
Sample Problem 14.2

A 20-kg projectile is moving with a velocity of 100 m/s when it explodes into 5 and 15-kg fragments. Immediately after the explosion, the fragments travel in the directions $\theta_A = 45^\circ$ and $\theta_B = 30^\circ$.

Determine the velocity of each fragment.

SOLUTION:

- Since there are no external forces, the linear momentum of the system is conserved.
- Write separate component equations for the conservation of linear momentum.
- Solve the equations simultaneously for the fragment velocities.
SOLUTION:

- Since there are no external forces, the linear momentum of the system is conserved.

- Write separate component equations for the conservation of linear momentum.

\[ x \text{ components:} \]

\[ y \text{ components:} \]

- Solve the equations simultaneously for the fragment velocities.

\[ v_A = 207 \text{ m/s} \quad v_B = 97.6 \text{ m/s} \]
Kinetic Energy

- Kinetic energy of a system of particles,

- Expressing the velocity in terms of the centroidal reference frame,

\[ \vec{v}_i = \vec{v}_G + \vec{v}_i' \]

- Kinetic energy is equal to kinetic energy of mass center plus kinetic energy relative to the centroidal frame.

- Principle of work and energy can be applied to each particle $P_i$,\n  \[ T_1 + U_{1\rightarrow2} = T_2 \]
  where $U_{1\rightarrow2}$ represents the work done by the internal forces $\vec{f}_{ij}$ and the resultant external force $\vec{F}_i$ acting on $P_i$.

- Principle of work and energy can be applied to the entire system by adding the kinetic energies of all particles and considering the work done by all external and internal forces.

- Although $\vec{f}_{ij}$ and $\vec{f}_{ji}$ are equal and opposite, the work of these forces will not, in general, cancel out.

- If the forces acting on the particles are conservative, the work is equal to the change in potential energy and
  \[ T_1 + V_1 = T_2 + V_2 \]
  which expresses the principle of conservation of energy for the system of particles.
Principle of Impulse and Momentum

- The momenta of the particles at time $t_1$ and the impulse of the forces from $t_1$ to $t_2$ form a system of vectors *equipollent* to the system of momenta of the particles at time $t_2$. 
Sample Problem 14.4

Ball $B$, of mass $m_B$, is suspended from a cord, of length $l$, attached to cart $A$, of mass $m_A$, which can roll freely on a frictionless horizontal tract. While the cart is at rest, the ball is given an initial velocity $v_0 = \sqrt{2gl}$.

Determine (a) the velocity of $B$ as it reaches its maximum elevation, and (b) the maximum vertical distance $h$ through which $B$ will rise.

SOLUTION:

- With no external horizontal forces, it follows from the impulse-momentum principle that the horizontal component of momentum is conserved. This relation can be solved for the velocity of $B$ at its maximum elevation.

- The conservation of energy principle can be applied to relate the initial kinetic energy to the maximum potential energy. The maximum vertical distance is determined from this relation.
SOLUTION:

- With no external horizontal forces, it follows from the impulse-momentum principle that the horizontal component of momentum is conserved. This relation can be solved for the velocity of \( B \) at its maximum elevation.

\[ \text{Velocity at positions 1 and 2 are} \]

\[ \begin{align*}
  v_{A,1} &= 0 \\
  v_{B,1} &= v_0 \\
  v_{B/A,2} &= 0 \\
  v_{B,2} &= v_{A,2}
\end{align*} \]

(velocity of \( B \) relative to \( A \) is zero at position 2)

\[ v_{A,2} = v_{B,2} = \frac{m_B}{m_A + m_B} v_0 \]
The conservation of energy principle can be applied to relate the initial kinetic energy to the maximum potential energy.

\[ T_1 + V_1 = T_2 + V_2 \]

Position 1 - Potential Energy: \( V_1 = m_A gl \)

\( \text{Kinetic Energy: } T_1 = \frac{1}{2} m_B v_0^2 \)

Position 2 - Potential Energy: \( V_2 = m_A gl + m_B gh \)

\( \text{Kinetic Energy: } T_2 = \frac{1}{2} (m_A + m_B) v_{A,2}^2 \)

\[ h = \frac{m_A}{m_A + m_B} \frac{v_0^2}{2g} \]
Sample Problem 14.5

Ball A has initial velocity $v_0 = 10 \text{ m/s}$ parallel to the axis of the table. It hits ball B and then ball C which are both at rest. Balls A and C hit the sides of the table squarely at A’ and C’ and ball B hits obliquely at B’.

Assuming perfectly elastic collisions, determine velocities $v_A$, $v_B$, and $v_C$ with which the balls hit the sides of the table.

SOLUTION:

- There are four unknowns: $v_A$, $v_{B,x}$, $v_{B,y}$, and $v_C$.
- Solution requires four equations: conservation principles for linear momentum (two component equations), angular momentum, and energy.
- Write the conservation equations in terms of the unknown velocities and solve simultaneously.
SOLUTION:

- There are four unknowns: $v_A$, $v_{B,x}$, $v_{B,y}$, and $v_C$.

\[ \vec{v}_A = v_A \hat{j} \]
\[ \vec{v}_B = v_{B,x} \hat{i} + v_{B,y} \hat{j} \]
\[ \vec{v}_C = v_C \hat{i} \]

- The conservation of momentum and energy equations,

\[ \vec{L}_1 + \sum \int \vec{F} dt = \vec{L}_2 \]
\[ mv_0 = mv_{B,x} + mv_C \quad 0 = mv_A - mv_{B,y} \]
\[ \vec{H}_{O,1} + \sum \int \vec{M}_O dt = \vec{H}_{O,2} \]
\[ -(2 \text{ ft})mv_0 = (8 \text{ ft})mv_A - (7 \text{ ft})mv_{B,y} - (3 \text{ ft})mv_C \]
\[ T_1 + V_1 = T_2 + V_2 \]
\[ \frac{1}{2}mv_0^2 = \frac{1}{2}mv_A^2 + \frac{1}{2}m(v_{B,x}^2 + v_{B,y}^2) + \frac{1}{2}mv_C^2 \]

Solving the first three equations in terms of $v_C$,
\[ v_A = v_{B,y} = 3v_C - 20 \quad v_{B,x} = 10 - v_C \]

Substituting into the energy equation,
\[ 2(3v_C - 20)^2 + (10 - v_C)^2 + v_C^2 = 100 \]
\[ 20v_C^2 - 260v_C + 800 = 0 \]

\[ v_A = 4 \text{ ft/s} \quad v_C = 8 \text{ ft/s} \]
\[ \vec{v}_B = (2\hat{i} - 4\hat{j}) \text{ ft/s} \quad v_B = 4.47 \text{ ft/s} \]
Variable Systems of Particles

• Kinetics principles established so far were derived for constant systems of particles, i.e., systems which neither gain nor lose particles.

• A large number of engineering applications require the consideration of variable systems of particles, e.g., hydraulic turbine, rocket engine, etc.

• For analyses, consider auxiliary systems which consist of the particles instantaneously within the system plus the particles that enter or leave the system during a short time interval. The auxiliary systems, thus defined, are constant systems of particles.
Steady Stream of Particles

- System consists of a steady stream of particles against a vane or through a duct.

- Define auxiliary system which includes particles which flow in and out over $\Delta t$.

- The auxiliary system is a constant system of particles over $\Delta t$.

$$\bar{L}_1 + \sum_{t_1}^{t_2} \int F \, dt = \bar{L}_2$$

$$[\sum m_i \bar{v}_i + (\Delta m)\bar{v}_A] + \sum \bar{F} \Delta t = [\sum m_i \bar{v}_i + (\Delta m)\bar{v}_B]$$

$$\sum \bar{F} = \frac{dm}{dt} (\bar{v}_B - \bar{v}_A)$$
Steady Stream of Particles. Applications

- Fluid Stream Diverted by Vane or Duct
- Fluid Flowing Through a Pipe
- Jet Engine
- Fan
- Helicopter
Streams Gaining or Losing Mass

- Define auxiliary system to include particles of mass \( m \) within system at time \( t \) plus the particles of mass \( \Delta m \) which enter the system over time interval \( \Delta t \).

- The auxiliary system is a constant system of particles.

\[
\vec{L}_1 + \sum_{t_1}^{t_2} \int_0 \vec{F} dt = \vec{L}_2
\]

\[
[m \vec{v} + (\Delta m)\vec{v}_a] + \sum \vec{F} \Delta t = (m + \Delta m)(\vec{v} + \Delta \vec{v})
\]

\[
\sum \vec{F} \Delta t = m \Delta \vec{v} + \Delta m(\vec{v} - \vec{v}_a) + (\Delta m)\Delta \vec{v}
\]

\[
\sum \vec{F} = m \frac{d\vec{v}}{dt} + \frac{dm}{dt} \vec{u}
\]

\[
m\vec{a} = \sum \vec{F} - \frac{dm}{dt} \vec{u}
\]
Sample Problem 14.6

Grain falls onto a chute at the rate of 240 lb/s. It hits the chute with a velocity of 20 ft/s and leaves with a velocity of 15 ft/s. The combined weight of the chute and the grain it carries is 600 lb with the center of gravity at $G$.

Determine the reactions at $C$ and $B$.

SOLUTION:

• Define a system consisting of the mass of grain on the chute plus the mass that is added and removed during the time interval $\Delta t$.

• Apply the principles of conservation of linear and angular momentum for three equations for the three unknown reactions.
SOLUTION:

- Define a system consisting of the mass of grain on the chute plus the mass that is added and removed during the time interval $\Delta t$.

- Apply the principles of conservation of linear and angular momentum for three equations for the three unknown reactions.

\[
\vec{L}_1 + \sum \int \vec{F} \, dt = \vec{L}_2
\]

\[
C_x \Delta t = (\Delta m)v_B \cos 10^\circ
\]

\[
-(\Delta m)v_A + (C_y - W + B)\Delta t = -(\Delta m)v_B \sin 10^\circ
\]

\[
\vec{H}_{C,1} + \sum \int \vec{M}_C \, dt = \vec{H}_{C,2}
\]

\[
-3(\Delta m)v_A + (-7W + 12B)\Delta t
\]

\[
= 6(\Delta m)v_B \cos 10^\circ - 12(\Delta m)v_B \sin 10^\circ
\]

Solve for $C_x$, $C_y$, and $B$ with

\[
\frac{\Delta m}{\Delta t} = \frac{240 \text{ lb/s}}{32.2 \text{ ft/s}^2} = 7.45 \text{ slug/s}
\]

\[
B = 423 \text{ lb} \quad \vec{C} = (110.1 \hat{i} + 307 \hat{j}) \text{ lb}
\]
The **effective force** of a particle \( P_i \) of a given system is the product \( m_i a_i \) of its mass \( m_i \) and its acceleration \( a_i \) with respect to a newtonian frame of reference centered at \( O \). The system of the external forces acting on the particles and the system of the effective forces of the particles are equipollent; i.e., both systems have the same resultant and the same moment resultant about \( O \):

\[
\sum_{i=1}^{n} F_i = \sum_{i=1}^{n} m_i a_i \quad \sum_{i=1}^{n} (r_i \times F_i) = \sum_{i=1}^{n} (r_i \times m_i a_i)
\]

The **linear momentum** \( L \) and the **angular momentum** \( H_o \) about point \( O \) are defined as

\[
L = \sum_{i=1}^{n} m_i v_i \quad H_o = \sum_{i=1}^{n} (r_i \times m_i v_i) \quad \sum F = \dot{L} \quad \sum M_o = \dot{H_o}
\]

This expresses that the resultant and the moment resultant about \( O \) of the external forces are, respectively, equal to the rates of change of the linear momentum and of the angular momentum about \( O \) of the system of particles.
Motion of the mass center of a system of particles

The mass center \( G \) of a system of particles is defined by a position vector \( \mathbf{r} \) which satisfies the equation

\[
m\mathbf{r} = \sum_{i=1}^{n} m_i \mathbf{r}_i
\]

where \( m \) represents the total mass \( \sum_{i}^{n} m_i \). Differentiating both members twice with respect to \( t \), we obtain

\[
\mathbf{L} = m\mathbf{\bar{v}}
\]
\[
\dot{\mathbf{L}} = m\mathbf{\bar{a}}
\]

where \( \mathbf{\bar{v}} \) and \( \mathbf{\bar{a}} \) are the velocity and acceleration of the mass center \( G \). Since \( \sum \mathbf{F} = \mathbf{L} \), we obtain \( \sum \mathbf{F} = m\mathbf{\bar{a}} \)

Therefore, the mass center of a system of particles moves as if the entire mass of the system and all the external forces were concentrated at that point.

\[
m\mathbf{\bar{v}} = \sum_{i=1}^{n} m_i \mathbf{\bar{v}}_i
\]
Angular momentum of a system of particles about its mass center

Consider the motion of the particles of a system with respect to a centroidal frame $Gx'y'z'$ attached to the mass center $G$ of the system and in translation with respect to the newtonian frame $Oxyz$. The angular momentum of the system about its mass center $G$ is defined as the sum of the moments about $G$ of the momenta $m_iv'_i$ of the particles in their motion relative to the frame $Gx'y'z'$. The same result is obtained by considering the moments about $G$ of the momenta $m_iv_i$ of the particles in their absolute motion. Therefore

$$H_G = \sum_{i=1}^{n} (\vec{r'}_i \times m_i\vec{v}_i) = \sum_{i=1}^{n} (\vec{r'}_i \times m_i\vec{v'}_i)$$
We can derive the relation

\[ \sum M_G = \dot{H}_G \]

which expresses that the moment resultant about \( G \) of the external forces is equal to the rate of change of the angular momentum about \( G \) of the system of particles. When no external force acts on a system of particles, the linear momentum \( \mathbf{L} \) and the angular momentum \( \mathbf{H}_o \) of the system are conserved. In problems involving central forces, the angular momentum of the system about the center of force \( O \) will also be conserved.